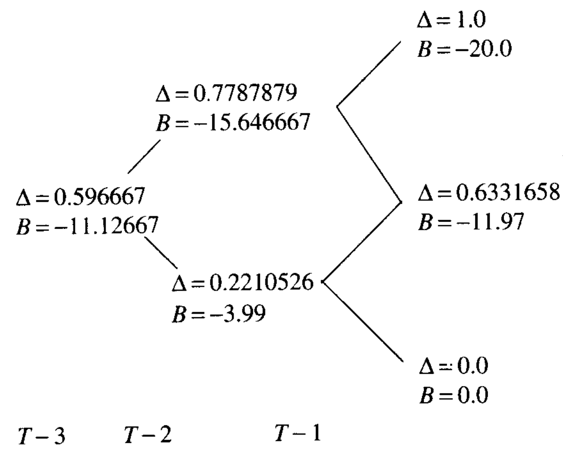


Figure 17.6 American call.

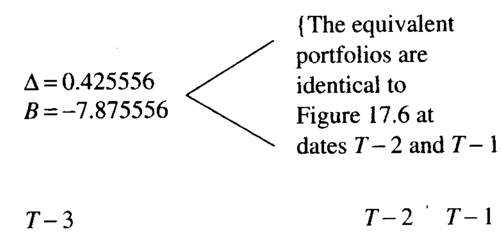


Figure 17.7 European call.

Use the American call values in the numerator of Equation (17.3) when solving for the number of shares needed to replicate American calls. Use European call values when solving for European deltas. Finally, use cum-dividend values of the stock in the denominator of Equation (17.3).

Here again, remember to use the appropriate (American or European) call values in the numerator in Equation (17.4) when solving for the amount to be invested in bonds as part of the call replication process.

Figure 17.6 depicts the time path of the equivalent portfolio for the American call. Figure 17.7 shows that only the time $T-3$ equivalent portfolio is different when one is examining European calls.

At each date, the payoffs of the equivalent portfolio are identical to the values of the call.

At time $T-3$ for an American call:

$$\Delta S + B = 0.596667(20) - 11.12667 = 0.806667 = C_{T-3}(\text{American})$$

$$0.596667(22) - 11.12667 = 2 = C_{T-2,u}\ (\text{American})$$

$$0.596667(19) - 11.12667 = 0.21 = C_{T-2,d}$$

At time $T-3$ for a European call:

$$\Delta S + B = 0.425556(20) - 7.875556 = 0.635556 = C_{T-3}(\text{European})$$

$$\nearrow\ 0.425556(22) - 7.875556 = 1.48667 = C_{T-2,u}\ (\text{European})$$

$$\searrow\ 0.425556(19) - 7.875556 = 0.21 = C_{T-2,d}$$

Additionally, if the stock has an uptick in the first period, then:

$$\Delta S + B = 0.7787879(22) - 15.646667 = 1.486667 = C_{T-2,u}$$

$$\nearrow\ 0.7787879(24.2) - 15.646667 = 3.2 = C_{T-1,uu}$$

$$\searrow\ 0.7787879(20.9) - 15.646667 = 0.63 = C_{T-1,ud}$$

Note that if the call is American, it is cheaper to replicate the call than it is to buy it. At time $T-2$, after an uptick, the American call must sell for its intrinsic value of \$2 to prevent arbitrage opportunities. Yet there will be no buyers for the call if its market price is \$2. An investor can instead buy 0.7787879 share of stock at \$22/share and borrow \$15.646667. The payoffs of this levered portfolio are identical to those of the call, and it is cheaper to buy the levered portfolio, so why pay \$2 for the call? The call owner will not find buyers for the call, and it will be rational for him to exercise it, thereby acquiring the stock for \$20, sell it for its market price of \$22, and then buy the equivalent portfolio. This point can be illustrated using an arbitrage table:

			Time $T-1$ (ex-dividend)	
	Time $T-2$		**Stock=\$23.2**	**Stock=\$19.9**
Exercise call; pay K	−20			
Sell stock acquired	+22			
Buy 0.7787879 share		Dividend	+0.7787879	+0.7787879
of stock at \$22/share	−17.133333	Sell stock	+18.067879	+15.497879
Borrow	+15.646667	Repay loan	−15.646667	−15.646667
	+0.513333		+3.2	+0.63

The table illustrates that if there is an uptick in the first period, the call holder will be better off at time $T-2$ by exercising the call and then using stocks and bonds to replicate it. The payoffs at time $T-1$ are the same regardless of whether he replicates or owns the call.

The student will be asked to verify that the remaining equivalent portfolios offer the same payoffs as the call at each date and state of the world (Problem 17.9 at the end of the chapter).

Finally, we must verify that the equivalent portfolio is self-financing at each date and state of the world. At time $T-3$ the American call replicator owns 0.596667 share of stock that sells for \$20/share and has borrowed \$11.126667. If the stock has a downtick in the first period, the new Δ is 0.2210526, and the new B is −3.99. The equivalent portfolio requires the sale of (0.596667−0.2210526=) 0.3756141 share of stock at \$19/share, providing proceeds of \$7.136667. This amount is exactly the amount needed to pay down the debt from \$11.126667 to \$3.99. Thus the equivalent portfolio for the American call is self-financing if the stock has a downtick.[14]

Now consider the equivalent portfolio for the European call at time $T-3$. It consists of 0.425556 share of stock and borrowing of \$7.875556. If the stock has an uptick in the first period, the equivalent portfolio then consists of 0.7787879 share of stock that sells for \$22/share and \$15.646667 in debt. The additional cash needed to buy the additional 0.3532323 share of stock at \$22/share is \$7.771111. However, because the riskless interest rate is 0%, the additional borrowing required to continue with the replication is (\$15.646667 – \$7.875556=) \$7.771111. Thus the equivalent portfolio is self-financing if the stock has an uptick in the first period.

If the stock has a downtick in the first period, the European Δ declines from 0.425556 to 0.2210526. The sale of 0.204503 share of stock at \$19/share provides \$3.885556. This equals the decline in required borrowing from the B value of \$7.875556 at time $T-3$ to the B value of 3.99 at time $T-2$.

Demonstrating that the equivalent portfolio is self-financing at all subsequent dates and outcomes is left as an exercise for the student (Problem 17.10 at the end of the chapter).

17.4 Puts

17.4.1 Simple Binomial Put Pricing

The derivation of the equations that define the single-period BOPM equivalent portfolio for a put is similar to the one for calls explained in Section 17.2.2. To start, define the pricing process for the underlying asset to be:

$$S_{T-1} \begin{cases} \nearrow S_{T,u} = (1+u)S_{T-1} \\ \searrow S_{T,d} = (1+d)S_{T-1} \end{cases}$$

As with calls, time T is the expiration date of a put option. If the stock rises, the put will be worth $P_{T,u}=\max(0, K-S_{T,u})$. If the stock declines in value at time T, the put is worth $P_{T,d}= \max(0, K-S_{T,d})$. Graphically, we have

$$P_{T-1} \begin{cases} \nearrow P_{T,u} = \max(0, K - S_{T,u}) = \max(0, K-(1+u)S_{T-1}) \\ \searrow P_{T,d} = \max(0, K - S_{T,d}) = \max(0, K-(1+d)S_{T-1}) \end{cases}$$

The portfolio making the same payoffs at time T as the put requires an investment in Δ shares of stock and B in riskless bonds (debt instruments). We will see that for a portfolio equivalent to a put, Δ will always be negative or zero (in other words, Δ is nonpositive) and B will always be positive or zero (nonnegative). This means that a put is equivalent to a portfolio consisting of a short position in stock and lending. We assume that the put replicator receives full use of the proceeds from the short sale of stock.

Graphically, the equivalent portfolio follows the same process as for calls:

$$\Delta S_{T-1} + B \begin{cases} \nearrow \Delta(1+u)S_{T-1} + (1+r)B = \Delta S_{T,u} + (1+r)B \\ \searrow \Delta(1+d)S_{T-1} + (1+r)B = \Delta S_{T,d} + (1+r)B \end{cases}$$

Now, equate the payoffs of the equivalent portfolio with the values of the put at time T:

$$\Delta(1+u)S_{T-1}+(1+r)B=P_{T,u}$$
$$\Delta(1+d)S_{T-1}+(1+r)B=P_{T,d}$$

These two equations are simultaneous equations with two unknowns, Δ and B. When we achieve our goal of finding the values of these unknowns, we will have defined a levered portfolio that always pays off the same as the put. Solving the system of simultaneous equations, we get:

$$\Delta=\frac{P_{T,u}-P_{T,d}}{(u-d)S_{T-1}}=\frac{P_{T,u}-P_{T,d}}{S_{T,u}-S_{T,d}};\quad \Delta\leq 0 \tag{17.20}$$

$$B=\frac{(1+u)P_{T,d}-(1+d)P_{T,u}}{(u-d)(1+r)};\quad B\geq 0 \tag{17.21}$$

Note that Equations (17.20) and (17.21) are identical to equations (17.1) and (17.2) except that put values are substituted for call values and that the signs for Δ and B have changed. The change in signs stems from differences in intrinsic value for the put versus the call.

The general formulation for the put Δ and B are:

$$\Delta=\frac{P_u-P_d}{(u-d)S}=\frac{P_u-P_d}{S_u-S_d} \tag{17.22}$$

$$B=\frac{(1+u)P_d-(1+d)P_u}{(u-d)(1+r)} \tag{17.23}$$

The values of Δ and B define the number of shares of stock to sell and the amount to lend to replicate a put. If the put and the debt–equity portfolio both offer exactly the same payoffs at time T, then the price of the put at time $T-1$ must equal the investment in the equivalent portfolio at time $T-1$:

$$P_{T-1}=\Delta S_{T-1}+B \tag{17.24}$$

Next, substitute the expressions for Δ [Equation (17.22)], and B [Equation (17.23)] into the expression for the put value [Equation (17.24)]. After simpifying, we get:

$$P_{T-1}=\frac{[(r-d)/(u-d)]P_{T,u}+[(u-r)/(u-d)]P_{T,d}}{1+r}$$

or

$$P_{T-1}=\frac{pP_{T,u}+(1-p)P_{T,d}}{1+r} \tag{17.25}$$

where

$$p=\frac{r-d}{u-d}\qquad\text{and}\qquad 1-p=\frac{u-r}{u-d}$$

The general BOPM equation for the value of a European put option is given by Equation (17.26):

$$P = \frac{pP_u + (1-p)P_d}{1+r} \tag{17.26}$$

There will be many cases in which Equation (17.26) yields a value for P that is less than intrinsic value. This is permissible for a European put. Indeed, recall from Chapter 16 that in-the-money European puts will frequently sell for less than $K-S$.

If we are valuing an American put, the BOPM formula is:

$$P = \max\left[K - S,\ \frac{pP_u + (1-p)P_d}{1+r}\right] \tag{17.27}$$

In other words, if the value of $(pP_u+(1-p)P_d)/(1+r)$ is less than $K-S$ (intrinsic value of the put), the American put will be worth $K-S$. Recall that by Propositions X of Chapter 16, an American put can never sell for less than its intrinsic value.

Whenever the American put sells for $K-S$ before expiration, the put owner will find it rational to exercise it early. By doing so, he will realize the intrinsic value of $K-S$ and also find that he can replicate the put's payoffs for a cost less than $K-S$.

Because of the possibility of early exercise, there is no "simple" formula to value an American put in a multiperiod BOPM framework. You must use the single-period put BOPM to solve for its value recursively, starting at the expiration date. In this way, you can identify the dates and situations in which the put will be exercised early.

17.4.2 A Numerical Example of Binomial Put Pricing

EXAMPLE 17.3 Consider a four-date, three-period world with time T representing the expiration date. The stock price at time $T-3$ is 60. If the stock rises, the size of the uptick is 10% per period. If the stock declines, the downtick per period is −5%. The riskless interest rate is 2% per period. The stock pays no dividends. Suppose there is a put option with a strike price of 65. Then:

a. Find the value of the stock for each possible outcome at every date.

b. Find the value of the European put for each possible outcome at every date.

c. Find the value of the American put for each possible outcome at every date.

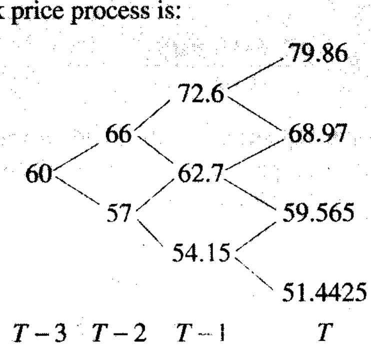
d. Find the composition of the equivalent portfolio for the European and American put for each possible outcome at every date.

e. Verify that the payoffs of each equivalent portfolio is identical for the put, regardless of whether there is an uptick or a downtick.

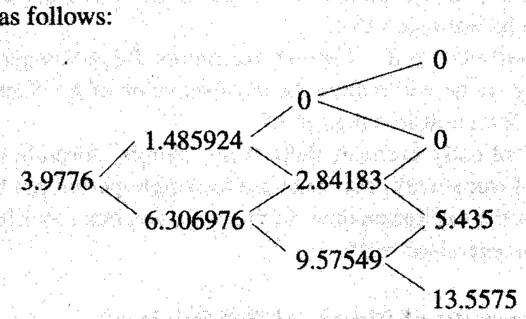
f. Verify that the equivalent portfolios are self-financing.

Use the recursive, single-period approach, computing the necessary values for each possible outcome at every date.

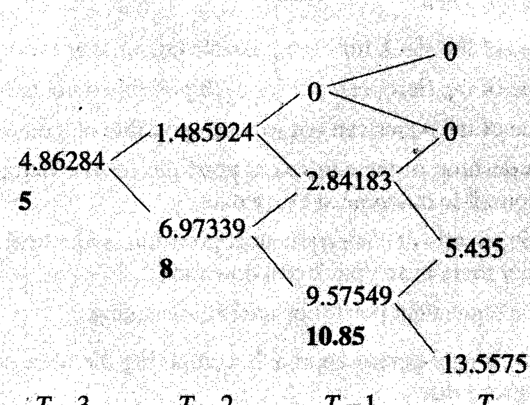
Solution: First, the stock price process is:



Next, we compute the value of p to be $(r-d)/(u-d)=(0.02-(-0.05))/(0.1-(-0.05)) = 0.466667$. Using Equation (17.26), recursively, we find that the values of the European puts are as follows:



Then we use Equation (17.27) to find the value of the American put. In the following diagram, the outcomes in which early exercise will occur contain two values: one for the value of $[pP_u + (1+p)P_d]/(1+r)$, and, since this value is less than the put's intrinsic value at that date, the put's intrinsic value at each time before T is presented in bold type. When $[pP_u + (1-p)P_d]/(1+r)$ is less than $K-S$, the value of the American put equals its intrinsic value.



This put should be exercised immediately, at time $T-3$. The put holder can exercise the put, realizing its intrinsic value of 5 and then replicate the put's payoffs for 4.86284.

For the European put, the composition of the equivalent portfolio at each date is defined by Equations (17.22) and (17.23):

		$\Delta = 0.0$ $B = 0.0$
	$\Delta = -0.2870535$ $B = 20.431458$	
$\Delta = -0.5356724$ $B - 36.117946$		$\Delta = -0.5778841$ $B = 39.075163$
	$\Delta = -0.7875626$ $B = 51.198042$	
		$\Delta = -1.0$ $B = 63.72549$
$T-3$	$T-2$	$T-1$

Thus, if there was an initial downtick in the first period, then at time $T-2$, $\Delta = (P_u - P_d)/(S_u - S_d) = (2.84183 - 9.57549)/(62.7 - 54.15) = -0.7875626$, and $B = [(1+u)P_d - (1+d)P_u]/[(u-d)(1+r)] = [(1.1)9.57549 - (0.95)2.84183]/[(0.1 - (-0.05))(1.02)] = 51.198042$.

For the American put, Equations (17.22) and (17.23) are again used to define the equivalent portfolio, but the American values for the puts are used in the numerators:

		$\Delta = 0.0$ $B = 0.0$
	$\Delta = -0.2870535$ $B = 20.431458$	
$\Delta = -0.7237862$ $B = 48.290014$		$\Delta = -0.5778841$ $B = 39.075163$
	$\Delta = -0.9366281$ $B = 60.361186$	
		$\Delta = -1.0$ $B = 63.72549$
$T-3$	$T-2$	$T-1$

For example, at time $T-3$, $\Delta = (P_u - P_d)/(S_u - S_d) = (1.485924 - 8)/(66 - 57) = -0.7237862$, and $B = [(1+u)P_d - (1+d)P_u]/[(u-d)(1+r)] = [(1.1)8 - (0.95)1.48592]/[(0.1 - (-0.05))(1.02)] = 48.290014$.

We now must verify that the payoffs of the equivalent portfolios equal those of the put. For the European put at time $T-3$, we have:

$$\begin{array}{ll} & (-0.5356724)(66)+(36.117946)(1.02) \\ & \quad = 1.485927 = P_{T-2,u} \\ \Delta S + B = (-0.5356724)(60) + 36.117946 & \\ \quad = 3.9776 = P_{T-3} & \\ & (-0.5356724)(57)+(36.117946)(1.02) \\ & \quad = 6.306978 = P_{T-2,d} \end{array}$$

Verification that the remaining equivalent portfolios pay off amounts identical to the European puts is left as an exercise for the student (Problem 17.11 at the end of the chapter).

For the American put at time $T-2$, given that there was an uptick in the first period, we have:

$$\begin{array}{ll} & (-0.2870535)(72.6)+(20.431458)(1.02) \\ & \quad = 0.0 = P_{T-1,uu} \\ \Delta S + B = (-0.2870535)(66) + 20.431458 & \\ \quad = 1.48593 = P_{T-2,u} & \\ & (-0.2870535)(62.7)+(20.431458)(1.02) \\ & \quad = 2.84183 = P_{T-1,ud} \end{array}$$

Verification that the remaining equivalent portfolios pay amounts that are identical to the American puts is again left as an exercise for the student (Problem 17.11).

Finally, we can verify that the equivalent portfolios are self-financing. For the European put, if there was a downtick in the first period, the equivalent portfolio at time $T-2$ consists of $\Delta=-0.7875626$ and $B=51.198042$. The required changes in the equivalent portfolio depend on whether there is an uptick or downtick at time $T-1$:

$\Delta = -0.7875626$, $B = 51.198042$

- Up: $\Delta = -0.5778841$, $B = 39{,}075163$ — Repurchase 0.2096785 share at \$62.70 per share. This costs \$13.146842. The original loan of \$51.198042 is worth (\$51.198042)(1.02) = \$52.222003. The reduction in the required loan to \$39.075163 is \$13.14684.
- Down: $\Delta = -1.0$, $B = 63.72549$ — Sell short an additional 0.2124374 share at \$54.15 per share. This provides \$11.503485. This is also the increase in required lending. You originally lend \$51.198042. Now you are owed \$52.222003. The requiréd equivalent B is \$63.72549.

For the American put, we will demonstrate that the equivalent portfolio will be self-financing only if there is an uptick.[15] At time $T-3$, the equivalent portfolio for the American put is $\Delta=-0.7237862$ and $B=48.290014$. If there is an uptick, the new equivalent portfolio is $\Delta=-0.2870535$ and $B=20.431458$. Thus, the replicator must repurchase 0.4367327 share of stock at \$66 per share, which requires \$28.824358. After interest, the new loan amount is (1.02)(48.290014)=\$49.255814. The required equivalent portfolio loan amount of \$20.431458 is \$28.824356 less than the outstanding loan balance, and this reduction in lending pays for the repurchase of shares that were originally sold short.

17.5 Portfolio Insurance and Dynamic Trading

Dynamic trading is synonymous with option replication using equivalent portfolios. Insuring portfolios with dynamic trading techniques has received considerable blame for the stock market "break" of October 1987. Since the foundations of insuring via dynamic trading lie in the BOPM, it seems appropriate to illustrate this investment technique now, and to show how it acted to accentuate market momentum (on the upside and then on the downside) and may indeed have contributed to the events of October 1987.

Portfolio insurance is a strategy used by traders who seek to eliminate the possibility of losses during some interval of time. Actually, users of this strategy can insure that portfolio returns will not fall below a minimum target rate of return, as long as that return is less than the riskless rate of return during the insurance horizon. For example, suppose the riskless rate of return is 10% per year, or 33.1% over the three-year insurance horizon. The insurer can guarantee that he will do no worse than earn a chosen return of $x\%$, where $x \leq 33.1\%$. If he wishes to avoid losses, he selects $x=0\%$. Depending on their strategies, other insurers might select values of x +10% (+3.228% per year), or −14.26% (−5% per year).

If, in our example $x=33.1\%$, the insurer would invest 100% of his portfolio's funds in riskless assets. Usually, $x<33.1\%$, in which case the insurer effectively sells some of his upside profits in exchange for truncating his minimum rate of return at $x\%$. If the return on the underlying asset is less than $x\%$, the insurer will earn the minimum rate of return. If the return on the underlying assets exceeds $x\%$, the insurer will participate in some of the increase in value. For example, suppose $x=0\%$ for the insurance horizon. Then, the insured portfolio earns 0% if the market return is negative or zero. However, if the market rises, the insured portfolio will not profit by the full extent of the rise. If the market were to rise by 50% over a three-year insurance horizon, an insured portfolio might earn only 40%. Thus, the portfolio insurer pays for insurance with reduced upside capture.

The easiest way to implement portfolio insurance is to purchase a European put on the underlying asset. A European put will usually be cheaper to purchase than an otherwise equivalent American put, so the insurer will be better off buying the former, with a time to expiration equal to his insurance horizon. Let us assume that European puts with a wide range of strike prices exist, with times to expiration long enough to meet an insurer's horizon, and that the insurer's portfolio and the underlying asset are identical. We know that deep in-the-money European puts will frequently sell for less than intrinsic value. Thus, given the current portfolio value, there will be some strike price for which a put will have zero time value. For now, this is the put of interest to us.

As an example, suppose an investor wants to create a portfolio of stocks that mimics the semiconductor index (SOX). Let the SOX spot price S_0 be 810. Suppose the value of the portfolio is \$5 million, and there exists a three-year European put on the SOX with a strike price of $K=830$ selling for $P=\$20$ (note that this put has no time value). The insurer must buy one put (on 100 shares of the SOX) for each hundred shares of the SOX purchased. Effectively, 100 shares of the SOX costs \$81,000, and one put costs \$2000. With \$5 million, the insurer can buy 5,000,000/83,000=60.24 units of the SOX, where each unit covers 100 shares (i.e., 100 shares of stock and 1 put). Each put on 100 shares costs \$2000, meaning that 60.24 puts will cost \$120,480. Thus, \$4,879,520 is invested in stocks, and \$120,480 in puts. We ignore transactions costs and dividends and assume the stock and puts are infinitely divisible.

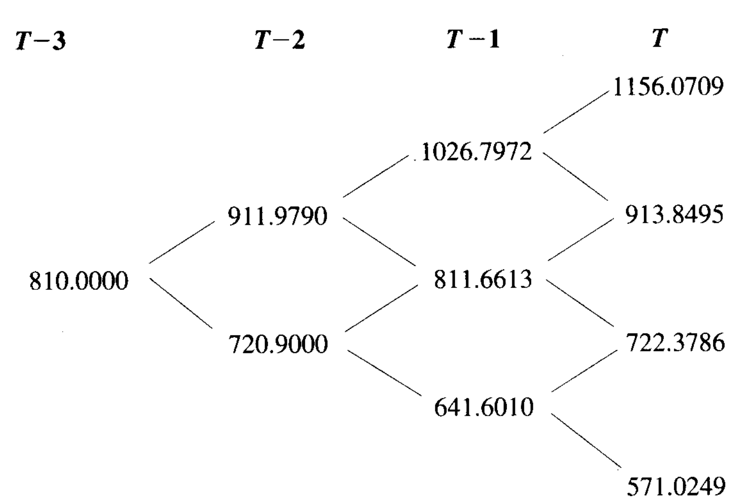
If $S_T \leq 830$ three years hence, the puts finish in the money. Because index options are cash settled at expiration, the put owner receives the in-the-money amount in cash. For example, if $S_T=798$, the put owner receives \$3200 for each put ($K-S_T=830-798=32$). He sells his 60.24 shares of the SOX for \$79,800 each and receives \$3200 for each of the 60.24 puts. The total cash received is \$5 million if $S_T \leq 830$.

If $S_T>830$, the puts expire worthless, and the insurer sells the portfolio of stocks for S_T. For example, if $S_T=861$, the cash received is $60.24 \times \$86,100 = \$5,186,664$. Note that the SOX index rose by 6.296%, but the insured portfolio's return was only 3.73%. The insured portfolio did not fully participate in the market's gains because the puts, which expired worthless, initially cost \$20.

Thus, investment in stock and the purchase of protective puts will insure a portfolio. Because the puts are purchased at time 0, the cost of the insurance is known up front.

Instead of buying puts, an insurer can replicate the puts by dynamically trading stocks and bonds. In other words, the insurer first invests 100% of his funds in the risky underlying asset (stock). Then he finds the equivalent portfolio for a put with the desired strike price and time to expiration. We know that the equivalent portfolio for a put is short stock ($-1 \leq \Delta \leq 0$) and lending ($B \geq 0$). The portfolio insurer need not sell stock short. Rather, the insurer sells the necessary stock from his portfolio and lends the proceeds.

For example, let $S=810$, $u=0.1259$, $d=-0.11$, $r=0.06$, $T=3$ years, and $K=830$. The stock pricing process is as follows:



The values of the European put, and the equivalent portfolios are as follows:

$T-3$	$T-2$	$T-1$	T
			$P_{T,uuu}=0.0000$
		$P_{T-1,uu}=0.0000$ $\Delta=0.0000$ $B=0.0000$	
	$P_{T-2,u}=7.4748$ $\Delta=-0.1318$ $B=127.7075$		$P_{T,uud}=0.0000$
$P_{T-3}=19.9858$ $\Delta=-0.2568$ $B=228.0292$		$P_{T-1,ud}=28.3629$ $\Delta=-0.5621$ $B=484.5792$	
	$P_{T-2,d}=56.5523$ $\Delta=-0.6648$ $B=535.8019$		$P_{T,udd}=107.6214$
		$P_{T-2,dd}=141.4179$ $\Delta=-1.0000$ $B=783.0189$	
			$P_{T,ddd}=258.9751$

As an exercise, verify that the equivalent portfolios pay off identical amounts to the puts and that the equivalent portfolio is self-financing [You should also use Equations (17.22) and (17.23) to check the values of Δ and B in the above diagram.]

Note that the put at time $T-3$ is selling just below its intrinsic value of 20. The difference between the put's price and its intrinsic value will create some slight deviations from perfect portfolio insurance in the subsequent analysis.

The dynamic trader who wishes to insure his portfolio will first invest \$4,879,602 in 60.242 units of the SOX at \$81,000/unit.[16] Each unit has 100 shares of the SOX. Then, he will replicate the purchase of 60.242 puts, using the equivalent portfolio at each date.

Thus, to replicate the long put position at time $T-3$, the insurer must sell 0.256844×60.242 units of the SOX at \$81,000/unit. This is because the put delta at time $T-3$ is -0.256844. This is \$1,253,296 of stock. He lends these proceeds and the remaining \$120,398[17] of his starting wealth to earn the riskless rate of return. Note that the amount lent, which is (\$1,253,296+120,398=) \$1,373,694, equals the value of B at time $T-3$ times 100 times 60.242. The portfolio equivalent to a long stock plus long put position at time $T-3$ is \$3,626,306 in stock and \$1,373,694 in riskless debt.

The same analysis proceeds at each date. The insurer will buy shares if Δ rises (Δ becomes less negative) or sell additional shares if Δ declines (Δ becomes more negative). The required loan at each date equals $B \times 100 \times 60.242$. The dynamic trading is self-financing, so that shares bought are financed by a reduction in lending, and shares sold are accompanied by an increase in lending. The time path of the equivalent portfolio for the long stock plus long put strategy is:

$$\$V_T = \$6,185,631(1.1259) = \$6,964,402$$

$$\$S = \$6,185,631 \quad \$B = \$0$$

$$\$S = \$4,769,638 \quad \$B = \$769,336$$

$$\begin{aligned} \$V_T &= \$6,185,631(0.89) \\ &= \$2,141,272(1.1259) \\ &\quad + 2,919,202(1.06) \\ &= \$5,505,212 \end{aligned}$$

$$\$S = \$3,626,306 \quad \$B = \$1,373,694$$

$$\$S = \$2,141,272 \quad \$B = \$2,919,202$$

$$\begin{aligned} \$V_T &= \$4,717,062(1.06) \\ &= \$2,141,272(0.89) \\ &\quad + 2,919,202(1.06) \\ &= \$5,000,086 \end{aligned}$$

$$\$S = \$1,455,750 \quad \$B = \$3,227,778$$

$$\$S = \$0.00 \quad \$B = \$4,717,062$$

$$\$V_T = \$4,717,062(1.06) = \$5,000,086$$

As stated earlier, because this put sold at time $T-3$ for less than its intrinsic value, some errors would emerge. They are most obvious when the stock has either 3 or 2 downticks, and the final portfolio value is \$5,000,086. Had the put initially sold for exactly \$20, perfect insurance on the downside would have been realized, and the terminal value of the portfolio would have been exactly \$5 million. The unobserved cost of insuring against a terminal value of \$5,000,086 is reduced upside capture if the SOX has 2 or 3 upticks.

If the market has three upticks, it will rise 42.725%. However, the insured portfolio rises only 39.288%. Similarly, the portfolio does not participate fully in the market rise if it has two upticks: the market rise is 12.821%, but the insured portfolio rises only 10.104%. Thus, the cost of insuring the portfolio is limited upside capture. Unlike the traded put (which has a one-time, up-front insurance cost equal to the price of the put), the dynamic trading approach pays for the insurance over time. Note also that the cost for insurance is unknown when dynamic trading is used to replicate the put. If market volatility increases (higher u and a more negative d), the degree of upside capture declines and there will also be greater risk that the final return will be less than the target minimum return.

Just before October 1987, an estimated \$80 billion in equity portfolios was insured by means of the dynamic trading approach just described. Traded puts were inadequate for insurance purposes for several reasons. First, their times to expiration were too short. Second, only American puts traded in 1987, which meant that the insurers would have to overpay for the protection that they wanted. Only since 1987 have European-style LEAPS begun trading, but even so, the liquidity of these long-term puts has been insufficient to satisfy the needs of many large investors. Finally, there has been, and currently is, an inadequate strike price range for the insurers. Thus, in 1987, dynamic trading was the preferred method for portfolio insurance.

For the first eight months of 1987, the market, measured by the Dow Jones Industrial Average, rose from 1896 to 2722 (a gain of almost 44%). Because put deltas become less negative as the market rises, insurers bought more stock as the market rose. Some observers believe that buying for insurance purposes helped to cause the stock market to rise to an unrealistic level by August 1987.

Then, in September and early October, the DJIA declined from a peak of 2722 on August 25, 1987 to 2505 on October 13. By the close of trading on October 16, the Dow had tumbled to 2247. As the foregoing diagrams illustrate, portfolio insurers sell stocks as the market declines. To reestablish their replicated protective puts, insurers had to sell enormous amounts of stock and stock index futures.[18] According to the Presidential Task Force on Market Mechanisms (1988), the so called the Brady Report, on three days, October 14, 15, and 16, insurers sold $3.7 billion in stock and stock index futures. On October 19 and 20, insurers sold an additional $5.2 billion and $2.1 billion more, respectively.

With such sustained selling by dynamic portfolio insurers, is it any wonder that the technique was blamed for exacerbating the decline? The Dow fell to 1738 on October 19, and the intraday low of 1709 was reached on October 20.

The purpose of placing the discussion of portfolio insurance and the crash in this chapter is the tie-in between dynamic trading and the BOPM. We close with two comments. First, there is still disagreement about just how much dynamic portfolio insurance contributed to the October 1987 crash. Large market declines occurred worldwide in October 1987, and portfolio insurance was nonexistent in many countries. Also, insurance-related selling on the crash days was only about 25% of the total NYSE volume. Obviously, others also sold stock and futures on those days. Finally, to the extent dynamic selling pressure lowered prices, other traders should have stepped up their buying to capitalize on the resulting unrealistic low prices.

Second, note that the unexpectedly high market volatility increased the "cost" of insurance. Shares had to be sold at prices well below those implied by the downtick returns. If a portfolio insurer who uses dynamic replication of options seriously misestimates volatility (the u and d parameters), the result will either be missed floors or forgone upside gains (see Rendleman and O'Brien, 1990). In response to the unexpectedly poor performance of insured portfolios, the size of the portfolio insurance industry greatly declined after October 1987. Jacobs (1999) claims that hundreds of billions of dollars of dynamic option replication still exists and could create financial crises in the future.

17.6 OTHER REFERENCES ON THE BINOMIAL OPTION PRICING MODEL AND DYNAMIC TRADING

The following references should be consulted for additional information about the binomial option pricing model (BOPM): Cox, Ross, and Rubinstein (1979), Rendleman and Bartter (1979), Rubinstein and Leland (1981), Jarrow and Rudd (1983, Chapter 13), Cox and Rubinstein (1985, Chapters 5 and 7), Boyle (1988), and Schroder (1988).

Papers on portfolio insurance using dynamic trading techniques include Rubinstein (1985, 1988), Abken (1987), O'Brien (1988a, 1988b), and Rendleman and O'Brien (1990).

Reports by the U.S. Commodity Futures Trading Commission (1988), the U.S. Securities and Exchange Commission (1988), and the Presidential Task Force on Market Mechanisms (1988) contain detailed information about the October 1987 stock market crash. More crash information is contained in Miller, Scholes, Malkiel, and Hawke (1988), Harris (1989), Kamphius, Kormendi, and Watson (1989), and Kleidon and Whaley (1992).

17.7 SUMMARY

A substantial portion of this chapter was devoted to the mechanics of applying the single-period BOPM to options. Besides ordinary European calls, we used the BOPM to value American calls on dividend paying stocks and European and American puts. The BOPM is extremely flexible. It can be used to value any kind of option and to incorporate possible early exercise as well as changing economic conditions (changing interest rates and changing stock volatility).

The single most important lesson to be learned is that if you know the binomial pricing process of the underlying stock, you can replicate any option, or indeed, any strategy covered in Chapter 15, with a portfolio of stocks and bonds. In a multiperiod world, the composition of this portfolio changes over time, thus the replicator must buy or sell stock every period[19]. However, the equivalent portfolio is self-financing, so that any stock purchase is accompanied by an increase in borrowing (for calls) or a decrease in lending (for puts).

Thus, we have an option pricing model, a method for valuing complicated options. To obtain accurate estimates of option values, however, the time to expiration must be carved into many intervals. The multiperiod BOPM for European calls can be cumbersome to use in this case, and the recursive single-period BOPM method is also time-consuming, even with a computer.

In the next chapter, we make some additional assumptions and obtain the Black–Scholes option pricing model. This model is easy to use, though at the cost of reduced flexibility.

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Notes

[1]The foundations for this chapter were developed in Cox, Ross, and Rubinstein (1979, pp. 229–263). Other articles that further developed the BOPM are listed at the end of the chapter.

[2]See Cox and Rubinstein (1985, Chapter 7) for applications of the BOPM to other stock pricing processes. A similar exposition by those authors is contained in Brenner (1983, Chapter 1). More advanced treatments of option valuation when the underlying asset follows different stochastic pricing processes are in Cox and Ross (1976) and Jarrow and Rudd (1983, Chapters 11 and 12).

[3]On one day, October 19, 1987, the Dow Jones Industrial Average fell 508 points, an unprecedented decline of 22.6% from the day before. Shortly after the crash, the size of stock portfolios covered by some form of insurance declined by about 50%. Portfolio insurance using replicated protective puts did not work as well as those who used the strategy had expected because the volatility of the market was greater than ever imagined. Still, option replication is widely employed today.

[4]$|B|$ is the absolute value of B. The statement means that the amount invested in the stock must exceed or equal the amount borrowed. This makes sense because we know that call values cannot be negative.

[5]Note that the formula for Δ may be read as the difference in call values divided by the difference in stock values. The time to expiration can be carved into as many subperiods (months, days, minutes, etc.) as you wish. As the length of the period shrinks toward zero, and the number of periods that make up the time to expiration increases to infinity, it can be shown that

$$\Delta = \frac{C_u - C_d}{S_u - S_d} \rightarrow \frac{\partial C}{\partial S}$$

which is the call's delta. It measures how much the call's value changes if the price of the underlying asset changes by a small amount, all else equal.

[6]This presentation can be rearranged to read $C-\Delta S=B$. This says that an investor who bought a fraction of one share of stock, denoted by Δ, and wrote one call, has created a riskless asset. Thus, Δ is frequently referred to as the call's **hedge ratio**, because Δ shares of stock plus one written call creates a riskless portfolio, or a "riskless hedge."

[7]One caveat must be made, however, if $C_{T-1} > \Delta S_{T-1}+B$. If we are dealing with American calls, the arbitrageur must be aware of the possibility of early exercise, since a call has been written as part of the arbitrage process. If the next date is not an ex-dividend date, there is no problem. A call owner will never exercise it early, and any that are irrationally early exercised would only mean greater profits for our arbitrageur. Nevertheless, we must be careful about writing a call in an attempt to arbitrage if the next day is an ex-date. This point will be covered in Section 17.3.3 on American calls on dividend-paying stocks.

[8]See Cox and Ross (1976) and Jarrow and Rudd (1983, pp. 88–95), and Jarrow and Turnbull (2000). Many authors call p and q "martingale probabilities".

[9]Recall that at any date, if the probability of a call on a non-dividend-paying stock finishing in the money is 1.0, then $\Delta=1.0$.

[10]Recall that $0!=1$.

[11]Dividend decreases occur much less frequently than do dividend increases.

[12]Since $F=Se^{rT}$, you should see the analogy between a futures or forward contract and a stock paying a continuous dividend.

[13]Note that the dividend amount if the stock rises is different from the amount if the stock falls. This will always be the case in the constant dividend yield model.

[14]Because the American call is exercised at time $T-2$ if there is an uptick in the stock's price, the self-financing demonstration does not apply.

[15]See also note 14. If the American put will be exercised early the next period, the self-financing property does not hold for the equivalent portfolio.

[16]The insurer must buy 60.242 units of the SOX and 60.242 puts with his \$5 million. One protective put is needed for each unit of the underlying stock purchased. The value of one put on 100 shares (P_{T-3}) is \$1998.58.

[17]The insurer began with \$5 million and invested \$4,879,602 in stocks. Thus, \$120,398 remained. Had the insurer used actual puts (rather than the replication process described here), he would have used the \$120,398 to buy 60.24 protective puts.

[18]Actually, most portfolio insurance is done with stock index futures, rather than the actual stock.

[19]The exception to this statement occurs when the call will finish out of the money with certainty (then $\Delta=0.0$) or in the money with certainty (then $\Delta=1.0$).

PROBLEMS

17.1 A stock sells for \$35 per share. One year from today, it will sell for either \$36/share or \$38/share. The riskless interest rate for the next year is 10%. Demonstrate how an arbitrage profit can be earned.

17.2 Let x and y be unknowns. Solve the following system of simultaneous equations:

$$5x+3y=13$$
$$3x+2y=9$$

17.3 In a binomial model framework, equate the payoffs of a call with the payoffs of a portfolio of Δ shares of stock and B in riskless debt. Algebraically solve the resulting set of simultaneous equations to obtain the definitions for Δ and B as they are given in Equations (17.1) and (17.2).

17.4 A stock currently sells for \$23/share. At the end of a single period, it can sell for either \$19/share or \$30/share. The riskless interest rate is 5%/period. You wish to find the value of a call with a strike price of 20. Set up the equations that equate the end-of-period payoffs of the call with the end-of-period payoffs of a portfolio of x shares of stock and y riskless bonds. The riskless bonds have a face value (maturity value) of \$1, and they mature at the end of the period. Solve the system of equations for x and y. Check your answers using Equations (17.1) and (17.2).

17.5 What is the "self-financing" requirement? Why is it necessary for the equivalent portfolio of stocks and bonds to be self-financing?

17.6 Use the data in Example 17.2, the multiperiod BOPM problem of Section 17.2.7, to find the value of a call with a strike price of 30.

17.7 Section 17.3.1 stated that if a stock has a constant dividend yield δ, it does not matter *when* the dividends are paid. What matters is *how many* ex-dividend dates there are prior to expiration. In the example in that section, the stock traded ex-dividend at times $T-4$ and $T-1$. Prepare the stock price path if the stock instead trades ex-dividend at times $T-3$ and $T-2$, and verify that the time T stock prices are the same as those in Figure 17.3.

17.8 Section 17.3.1 presented a two-period example of a stock paying a constant dividend yield:

$$
\begin{tikzcd}
 & & & 30.93468 \\
 & 28.69636 \arrow[r] & 28.12243 \arrow[ur] \arrow[dr] & \\
26.0876 \arrow[ur] \arrow[dr] & & & 26.71631 \\
 & 24.78322 \arrow[r] & 24.28756 \arrow[ur] \arrow[dr] & \\
 & & & 23.07318 \\
T-2 & T-1 & & T
\end{tikzcd}
$$

Assume that $r=0\%$. If the stock has a downtick at time $T-1$, find the composition of the equivalent portfolio at that time. Verify that the payoffs of that equivalent portfolio at time T are identical to $C_{T,du}$ and $C_{T,dd}$. Also verify that the equivalent portfolio was self-financing when moving from time $T-2$ to $T-1$.

17.9 In the examples in Sections 17.3.2 and 17.3.3 (Figures 17.5 and 17.6), verify that the equivalent portfolios have the same payoffs as the call at each date and state of the world. Be sure to verify this property for both upticks and downticks.

17.10 In the example in Section 17.3.2 (Figure 17.5), verify that the equivalent portfolio for the European call is self-financing at every date and state of the world. In the text, this was demonstrated for the passage of time from time $T-3$ to time $T-2$ at the end of section 17.3.3. Be sure to verify this property for both upticks and downticks.

17.11

a. Verify that all the equivalent portfolios pay off amounts that are identical to the puts at every date and every state of the world, for the example given in Section 17.4.2. Do this for both American and European puts.

b. Verify that the equivalent portfolios for the European put example of Section 17.4.2. are self-financing.

17.12 A stock currently sells for \$61.20. A call exists with a strike price of 60 and has four months until expiration. The stock can either rise by 2% each month or fall by 1% each month. The riskless interest rate is 0.5%/month. A dividend of \$1.50 per share will be paid two months hence (i.e., at time $T-2$). For each date and state of the world:

a. Plot out the time path of possible stock prices.

b. Recursively solve for the value of a European call.

c. Find the composition of the equivalent portfolio.

d. Pick one node, and verify that the payoffs of the equivalent portfolio are identical to those of the call.

e. Pick one node, and verify that the equivalent portfolio is self-financing.

17.13

a. Answer parts a–d of Problem 17.12, for an American call.

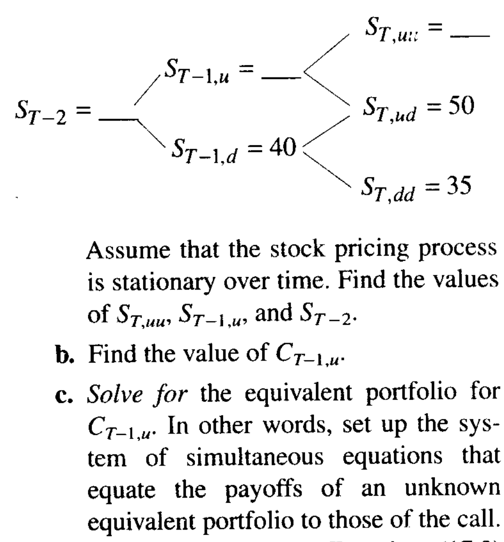
b. Assume a risk-neutral world. What is the probability of the stock having an uptick and a downtick? What is the probability that the stock will be exercised early at time $T-3$? Find the probability of early exercise at each subsequent date.

17.14 In the context of the BOPM, discuss how American puts are priced. In particular, focus on the possibility and rationality of early exercise.

17.15 In the context of the BOPM, discuss how American calls on dividend-paying stocks are priced. In particular, focus on the possibility and rationality of early exercise. When and how does exercising early dominate the strategy of holding on to the call?

17.16

a. Consider the problem posed as a quiz in Section 17.1. Suppose that the data are only part of a larger pricing process as follows:



Assume that the stock pricing process is stationary over time. Find the values of $S_{T,uu}$, $S_{T-1,u}$, and S_{T-2}.

b. Find the value of $C_{T-1,u}$.

c. *Solve for* the equivalent portfolio for $C_{T-1,u}$. In other words, set up the system of simultaneous equations that equate the payoffs of an unknown equivalent portfolio to those of the call. Solve for Δ and B. Use Equations (17.3) and (17.4) to check your answer. Verify that the payoffs of the equivalent portfolio are identical to those of the call.

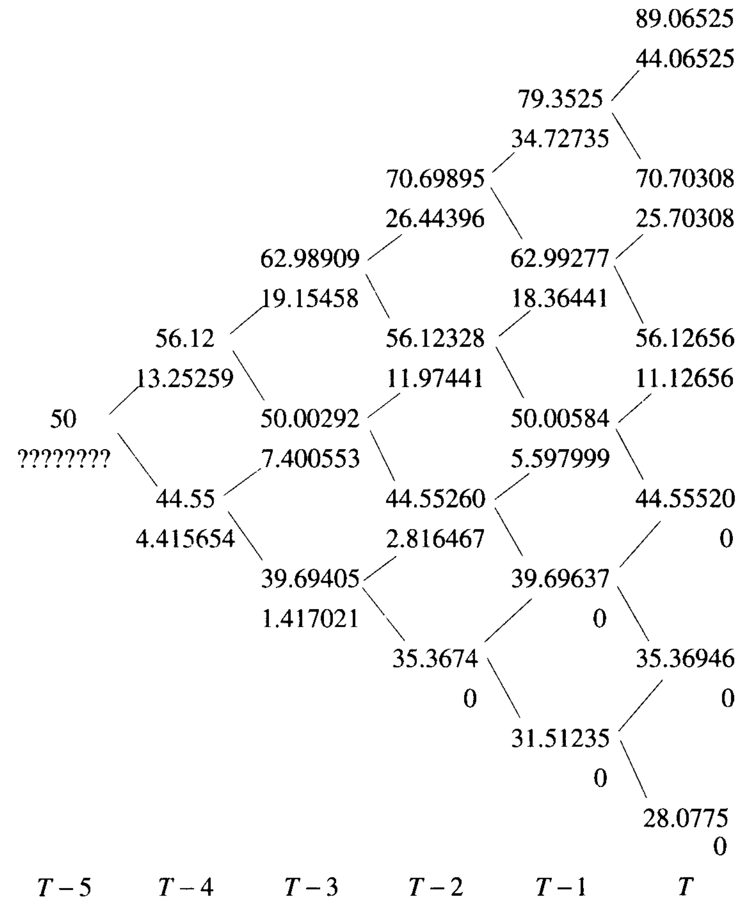
d. Use the multiperiod BOPM to find the value of C_{T-2}.

e. Use the single-period BOPM, and the two possible call values at time T–1 to find the value of C_{T-2}.

f. Find the equivalent portfolio for C_{T-2}. Verify that the payoffs of the equivalent portfolio are identical to those of the call. Verify that the equivalent portfolio is self-financing.

17.17 Use the multiperiod BOPM to estimate the value of a call that has four months to expiration and a strike price of 30. The stock sells for \$35.50/share. The size of an uptick is 10%/month, and the size of a downtick is 7%/month. The riskless interest rate is 0.583%/month.

17.18



The accompanying tree diagram is for a stock following a binomial pricing process. The risk-free interest rate is 0.84% per period. The upper number in each "box" is the stock price. The lower number is the price of a call. Time T is the expiration date of the call.

a. Compute the value of the call at time $T-5$.

b. Suppose the stock has had two upticks, so the stock sells for $62.98909 at time $T-3$. Find the composition of the equivalent portfolio of stocks and bonds at that "node." Then verify that regardless of whether there is an uptick or a downtick at time $T-2$, the payoffs of the equivalent portfolio and the call are the same.

c. Now, suppose time T is an ex-dividend date, and the dividend amount is $6 per share. Suppose the stock is selling for $62.99277 at time T–1. The ex-dividend prices at time T will be either $64.70308 or $50.12656. Will an American call be exercised early at time T–1? Why or why not?

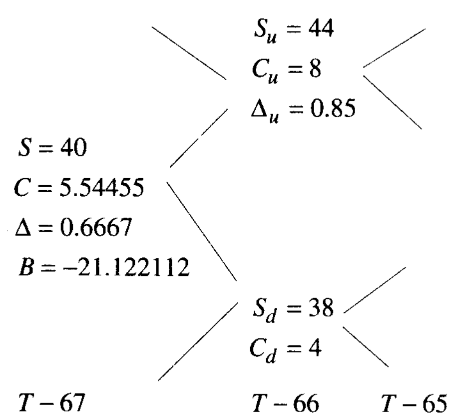
17.19 Today's stock price is $100. At the end of the upcoming single period, the stock will sell, ex-dividend, for either $140 or $65. The dividend amount is \$10. The riskless interest rate for the single period is 10%.

a. Find the theoretical value of an **American** call with a strike price of 90; the call expires at the end of the single period.

b. If the market price of the call at time T–1 is actually \$12, then clearly and thoroughly demonstrate how an arbitrage profit could be earned, given your answer to part a.

17.20 The diagram shows some data as part of a lengthy binomial process. The riskless interest rate is 1% per period.

a. Verify that the payoffs of the call at time $T-66$ are the same as the payoffs of the equivalent portfolio.



b. Compute B_u at time $T-66$, when $S_u = 44$. (Hint: the equivalent portfolio is self financing).

CHAPTER 18

Continuous Time Option Pricing Models

The binomial option pricing model (BOPM), produces reasonably accurate option values if the user has accurate beliefs about the values of u and d. Also, the BOPM is extremely flexible. For example, the BOPM can be used to value both European and American puts and calls, either with or without dividend payments. The possibility of early exercise can be accounted for in each period, and the values of u, d, and r can change over time. The primary drawback of the BOPM is that a computer must be used to estimate option values when the time to maturity is carved into many small periods.

18.1 The Black–Scholes Option Pricing Model

Under certain assumptions and when time to maturity is carved into an infinite number of subintervals, the BOPM will converge to the option pricing model attributed to Fischer Black and Myron Scholes. The Black–Scholes call option pricing model (henceforth, the BSOPM) provides an explicit solution to the problem of option pricing.[1] By "explicit," we mean that an equation is obtained.

Although the BSOPM is relatively simple to use, it is important to be aware that the model can accurately value only European options, or American calls on non-dividend-paying stocks. As the possibility of early exercise becomes more likely, the BSOPM produces increasingly inaccurate values. Moreover, an important assumption of the BSOPM is that the wanderings of the stock price through time follow one particular type of pricing process, which cannot change over time.

In this chapter, we first present a set of sufficient assumptions that will lead to the derivation of the BSOPM. Then, the model itself is stated, and a numerical example is shown in detail. Following that, we examine a few details concerning the assumed stochastic process guiding the value of the underlying asset. It is important to realize that a model generally will produce useful results only if the assumptions behind it are realistic. If the assumptions are violated in the real world, the model will frequently provide poor predictions.

18.1.1 Assumptions Behind the Black–Scholes Option Pricing Model

Although other sets of assumptions have been used to derive the BSOPM, the following assumption list is sufficient.

1. Capital markets are perfect. That is, there are no transactions costs or taxes. There are no short selling constraints, and investors get full use of the proceeds from short sales. All assets are infinitely divisible.
2. All investors can borrow and lend at the same riskless interest rate, which is constant over the life of the option.[2]
3. The stock pays no dividends. This means that the model can value European or American calls, since the latter will be exercised only at the expiration date, if and only if the call finishes in the money.
4. Markets are always open, and trading is continuous.
5. The wanderings of the stock price through time follow the rules of a specific type of stochastic process called a geometric Brownian motion. This pricing process is discussed in the chapter appendix.

18.1.2 A Quick Discussion of the Importance of Lognormal Returns

If the stock price follows a geometric Brownian motion, then the distribution of the stock's returns will be lognormally distributed.[3] The chapter appendix contains additional material on Brownian motion and returns distributions.

Lognormal returns are realistic for two reasons. First, if returns are lognormally distributed, the lowest possible return in any period is –100%. In contrast, if returns are normally distributed, there is some probability that returns will be less than –100%. Second, lognormal returns distributions are "positively skewed," that is, skewed to the right. This is realistic because while the lowest return in any period is –100%, the highest return will likely be in excess of 100%, particularly when measured over a year. Thus, a realistic depiction of a stock's returns distribution would have a minimum return of –100% and a maximum return well beyond +100%. The longer the time interval under consideration, the more valid the latter statement becomes. Therefore, annual returns will be more positively skewed than monthly returns, and monthly returns will be more skewed than daily returns. Returns distributions will not be symmetric. They will be skewed to the right.

18.1.3 The Variance of the Stock's Return Is Proportional to Time

Two comments must be made about the implications of this lognormal distribution assumption. First, although the price of the underlying asset (such as a stock) may be a function of its expected rate of return, the value of an option predicted by the BSOPM is independent of the expected return. In other words, all else equal, the value of an option on a stock that has a high expected rate of return is the same as the value of an option on an otherwise equivalent stock with a low expected rate of return.

Second, if the stock price follows a geometric Brownian motion process, the variance of the stock's returns is proportional to time. Correspondingly, the standard deviation of the stock's returns is proportional to the square root of time. Thus, define a subscript, l, to denote a long period of time, and define another subscript, s, to denote a short period of time. Then,

$$\sigma_l^2 = \left(\frac{l}{s}\right)\sigma_s^2 \tag{18.1}$$

EXAMPLE 18.1 If $s = 1$ day and $l = 1$ week, then

$$\sigma^2_{\text{weekly}} = (7)\sigma^2_{\text{daily}}$$

This means that the variance of weekly returns is seven times greater than the variance of daily returns. If $s = 1$ month and $l = 1$ year, the variance of yearly returns is 12 times the variance of monthly returns:

$$\sigma^2_{\text{yearly}} = (12)\sigma^2_{\text{monthly}}$$

The corresponding relationship for standard deviations is:

$$\sigma_l = \sqrt{\frac{l}{s}}\,\sigma_s \qquad (18.2)$$

The standard deviation of a stock's returns is proportional to the square root of time. For example, if $s = 1$ week, and $l = 1$ year, then

$$\sigma_{\text{yearly}} \sqrt{52}\sigma_{\text{weekly}} = 7.2111\sigma_{\text{weekly}}$$

This says that if the stock follows the stochastic pricing process that has been described, the standard deviation of a stock's yearly returns distribution will be 7.2111 times the standard deviation of its weekly returns distribution.

where σ^2_l is the variance of the stock's returns measured over the longer interval of time, σ^2_s is the variance of the stock's returns measured over the shorter period of time, and l/s is the number of short intervals in a long interval.

Perhaps an analogy will help make this important concept clear. Consider a person who starts to stagger along a straight line. Place him at point zero. After one day, he will have staggered back and forth, but will likely not have strayed too far from point zero. After one month, he may have wandered much further from point zero. After a year, he may be miles from it.

A stock's price behaves like this person. If today's stock price is $100/share, and it follows the pricing process described, then after one day has passed, it may be priced between $98 and $103 per share. After one month, it may sell between $90 and $115 per share. One year hence, it may sell for as low as $60 and as high as $160. The variance of the returns distribution grows larger as longer time intervals are considered. The variance of the annual returns distribution should be 365 times as great as the variance of the daily returns distribution. Figure 18.1 depicts the nature of the returns distributions as longer time intervals are considered.

18.2 The Black–Scholes Option Pricing Model and a Detailed Numerical Example

Under the preceding assumptions, the value of a European call, or an American call on a non-dividend-paying stock, is as follows:[4]

$$C = SN(d_1) - Ke^{-rT}N(d_2) \qquad (18.3)$$

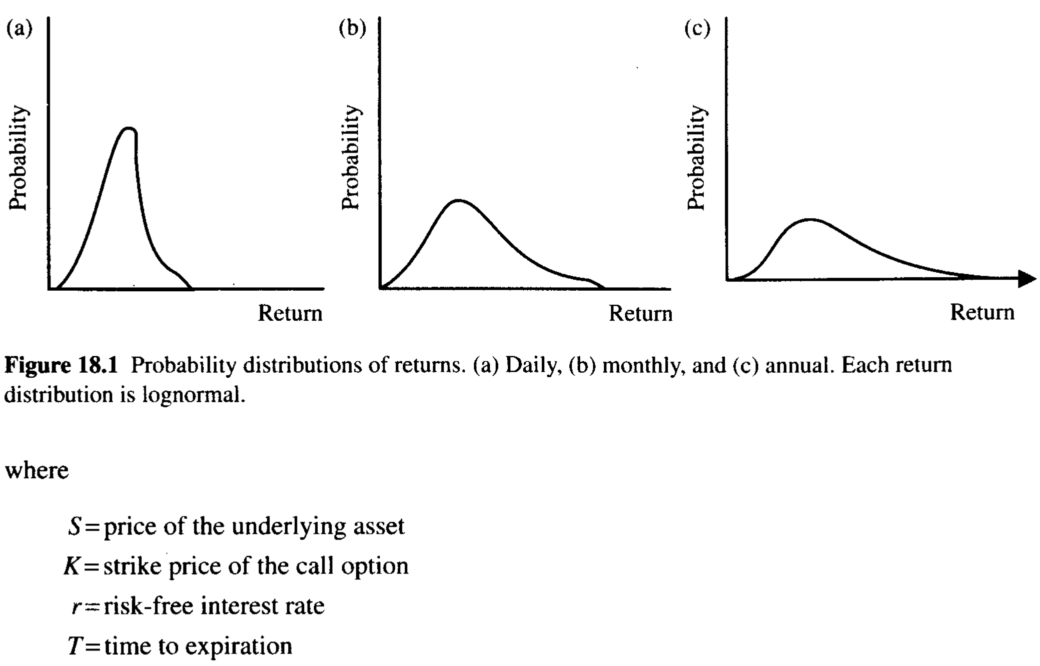


Figure 18.1 Probability distributions of returns. (a) Daily, (b) monthly, and (c) annual. Each return distribution is lognormal.

where

S = price of the underlying asset
K = strike price of the call option
r = risk-free interest rate
T = time to expiration
$N(d)$ = the cumulative standard normal distribution function[5]

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

σ = standard deviation of the underlying asset's returns
$\ln(S/K)$ = natural logarithm of S/K
e^{-rT} = exponential function of $-rT$. Thus, e^{-rT} is the present value factor when r is continuously compounded for time period T. The present value of K, therefore, is Ke^{-rT}.

An important aspect of the BSOPM is that both r and σ are assumed to be known and constant. In the BSOPM, r is generally stated as an annual interest rate; T, the time to expiration, is stated in years, and σ should be defined as the standard deviation of the stock's annual returns. However, any consistent interval of time can be used. For example, you may wish to define r as the monthly riskless interest rate, T as the number of months until expiration, and σ as the standard deviation of the stock's monthly returns.

For the value of r, some users employ the bond equivalent yield[6] on Treasury bills that mature near the option's expiration date. Other users believe that a more realistic rate is the Eurodollar interest rate, which incorporates a small default risk premium. For valuing options that expire more than one year in the future, r is typically the yield on the zero-coupon Treasury strip that matures on a date near the option's expiration date.

You might be surprised to see that the expected return on the stock does not appear in the formula. The value of an option is independent of the expected return of the stock.[7] However, the volatility of the stock returns, σ, is an important determinant of an option's value. In addition, σ is the only determinant of the call option value that is not directly observable. Users of the BSOPM must estimate σ, just as users of the BOPM had to estimate values for u and d.

Finally, note that there are no variables in the BSOPM that reflect the level of risk aversion in the economy. Under the stated assumptions, option values are invariant to investors' attitudes to risk.

EXAMPLE 18.2 Example of the BSOPM: A stock is currently selling for 58.875. The riskless interest rate is 8% per year. Estimate the value of a call with a strike price of 60 and a time to expiration of three months. The standard deviation of the stock's annual returns is 0.22. Thus, $S=58.875$, $K=60$, $T=0.25$ year, $r=0.08$/year, $\sigma=0.22$/year, and $\sigma^2=0.0484$. To find the Black–Scholes call option price, first calculate:

$$d_1 = \frac{\ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(58.875/60)+(0.08+0.0484/2)0.25}{0.22\sqrt{0.25}}$$

$$= \frac{\ln(0.98125)+0.02605}{(0.22)(0.5)} = \frac{-0.018928+0.02605}{0.11} = 0.064745$$

and

$$d_2 = d_1 - \sigma\sqrt{T} = 0.064745 - 0.22\sqrt{0.25} = 0.064745 - 0.11 = -0.045255$$

Next, values for $N(d_1)$ and $N(d_2)$ are needed, and they are found by using the cumulative standard normal distribution function table found in the appendix to this chapter[8]:

$$N(d_1) = 0.525812$$

and

$$N(d_2) = 0.481952$$

The discounted strike price is given by:

$$Ke^{-rT} = 60e^{-(0.08)(0.25)} = (60)e^{-0.02} = (60)(0.9802) = 58.81192$$

Finally, recall that the stock price is 58.875. Therefore the call option price is:

$$C = SN(d_1) - Ke^{-rT}N(d_2) = 58.875(0.525812) - 58.81192(0.481952) = 2.6127$$

This is the theoretical value of this call, and its accuracy depends critically on the estimate of σ. Later in this chapter, methods for estimating σ will be presented.

FinCAD provides the same solution, as shown in Figure 18.2. To generate the same solution for the call value, it is necessary to enter the value date as "today()" and the expiration (expiry) date as "today()+(365/4)." In addition, the annually compounded interest rate is exp(0.08)−1=8.3287068%. The meaning of the Greeks (delta, gamma, theta, vega, and rho) will be explained in the next chapter.

AaBS			
Underlying price	58.875		
Exercise price	60		
Expiry date	24-Jun-98		=TODAY()+(365/4)
Value (settlement) date	3/25/98		=TODAY()
Volatility	0.22		
Risk free interest rate	0.083287068		=exp(0.08)-1
Option type	1	Call	
Statistic	1	fair value	
Discounting method	1	annually compounded rate	
Accrual method	1	actual/ 365 (fixed)	
Fair value	**2.612634856**		1
Delta	**0.5258117**		2
Gamma	**0.061471836**		3
Theta	**-0.020339844**		4
Vega	**11.71927424**		5
Rho	**6.541324504**		6

Figure 18.2 The FinCAD function aaBS can be used to determine Black–Scholes prices for call and put options.

18.3 An Intuitive Look at the Black–Scholes Option Pricing Model

By assuming a world of certainty, we can gain an insightful view of the structure of the BSOPM. In such a world, all assets, including stock and all options, must be priced to yield the same riskless rate of return. If any asset was priced to yield more (and in a certain world, everyone would know this), then many investors would rush to buy the asset until its price rose enough to provide the same riskless return as all other assets. If any asset was priced so that its known rate of return was less, no investor would buy it, and its price would decline.

In a certain world, if investors knew that a call was going to finish out of the money, $S_T \leq K$, then the price of the call at every date would always be zero. On the other hand, suppose that investors know that a call will finish in the money. At expiration, everyone knows that C_T will sell for its intrinsic value, $S_T - K$. Because the call must be priced to yield the riskless return, today's price will equal the present value of C_T. In other words, $C_0 = [S_T - K]\,(1+r)^{-T}$, given that $S_T > K$. The discount rate, r, is the riskless interest rate.

In addition, the stock must also be priced to yield a riskless rate of return, r. Thus $S_0 = S_T(1+r)^{-T}$. This is equivalent to saying that $S_T = S_0(1+r)^T$. Substitute this into today's call value, and

$$C_0 = [S_0(1+r)^T - K](1+r)^{-T}$$

or

$$C_0 = S_0 - K(1+r)^{-T}$$

Compare this value of a call with the BSOPM. To convert the call value just given to the Black–Scholes call value, simply multiply S_0 by $N(d_1)$ and multiply $K(1+r)^{-T}$ by $N(d_2)$. Smith (1976, footnotes 20 and 22), and Jarrow and Rudd (1983, pp. 93–94) point out an interesting feature of the Black–Scholes model. They explain that if the Black–Scholes model is the correct option pricing model to use, the term $N(d_2)$ is the probability that the call will finish in the money in a risk-neutral world.[9]

18.4 The Black–Scholes Option Pricing Model and European Put Prices

One easy approach to estimate the value of a European put on a non-dividend-paying stock starts by calculating a call option price using the basic BSOPM. Then, a put value is obtained by using the basic put–call parity theorem. For example, in Section 18.2 we computed the theoretical value of a call with the following inputs:

$S = 58.875$

$K = 60$

$T = 0.25$ year

$r = 0.08$

$\sigma = 0.22$

Using the BSOPM, we found that the value of a European call is \$2.6127.

The basic put–call parity theorem is Proposition XI in Chapter 16:

$$C - P = S - K(1+r)^{-T}$$

If interest is compounded continuously, put–call parity can be restated as follows:

$$C - P = S - Ke^{-rT} \tag{18.4}$$

Rearranging, and solving for P:

$$P = C - S + Ke^{-rT} = 2.6127 - 58.875 + 60e^{-(0.08)(0.25)} = \$2.5496$$

Alternatively, the BSOPM formula can be restated to value European puts on non-dividend-paying stocks by using the continuous time version of the put–call parity theorem. That is, we start with the BSOPM

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

and rearrange the put–call parity proposition to be

$$C = P + S - Ke^{-rT} \tag{18.5}$$

This yields, after substituting Equation (18.5) into the BSOPM,

$$SN(d_1) - Ke^{-rT}N(d_2) = P + S - Ke^{-rT} \tag{18.6}$$

Solving Equation (18.6) for P results in

$$P = SN(d_1) - S + Ke^{-rT} - Ke^{-rT}N(d_2)$$

or

$$P = S[N(d_1) - 1] - Ke^{-rT}[N(d_2) - 1] \tag{18.7}$$

Note that in Equation (18.7), the terms $[N(d_1) - 1]$ and $[N(d_2) - 1]$ are negative because the area under the normal density function has a maximum of 1. Thus, Equation (18.7) can be restated as follows:

$$P = Ke^{-rT}[1 - N(d_2)] - S\,[1 - N(d_1)]$$

In addition, because of the symmetry of the normal density function, that is, because $[1 - N(d)] = N(-d)$, the put price can also be expressed as follows:

$$P = Ke^{-rT}N(-d_2) - SN(-d_1) \tag{18.8}$$

Now, let's use Equation (18.8) to value the put directly. Recall that in Example 18.2 we found that $d_1 = 0.064745$ and $N(d_1) = 0.525812$. Therefore, $1 - N(d_1) = N(-d_1) = 0.474188$. Furthermore, since $d_2 = -0.04525$ and $N(d_2) = 0.481952$, we have $1 - N(d_2) = N(-d_2) = 0.518048$. The value of the put equals:

$$P = Ke^{-rT}N(-d_2) - S\,N(-d_1) = (58.81192 \times 0.518048) - (58.875 \times 0.474188) = 2.5496$$

Note that this is the same value that we estimated by using the BSOPM call formula along with the put–call parity proposition. Intuitively, this makes sense because the put–call parity proposition was used to derive Equation (18.8).

18.5 Two Handy Extensions of the Black–Scholes Option Pricing Model

As you know, the BSOPM is designed to provide prices for European options on a security that does not pay dividends. Financial economists have modified and extended the BSOPM to allow its use to value options on securities of other types. We now turn our attention to an examination of two extensions to the BSOPM. These two variants of the BSOPM value European calls on stocks that will trade ex-dividend before the expiration date. Note that these two variants could also be used to value American calls if the user is highly confident that the call will not be exercised before expiration.

18.5.1 Variant One: The Black–Scholes Option Pricing Model on Securities Paying Known, Discrete Dividends

If the stock will trade ex-dividend before expiration, then the holder of a European call owns an option only on the stock price at expiration because he has no claim on dividends before the expiration date. The stock's theoretical value at expiration is the present value of all dividends after the expiration date. However, today's stock price is the present value of all dividends after today. Thus, the difference between the stock price today and the stock price at expiration is the present value of dividends received between tomorrow and the option expiration date. Therefore, a European call option on a dividend paying stock is really a call option on an asset price adjusted for the present value of all dividends received after today and prior to expiration.

To use this variant of the BSOPM, define

$$S^* = S - PV(\text{divs}) = S - \sum_{i=1}^{N} \text{div}_i (1+r)^{-t_i}$$

That is, S^* is today's stock price minus the present value of all dividends received after today and prior to the option expiration date. In this notation, div_i is the dividend amount at time t_i, and T is

---

EXAMPLE 18.3 Given the following information, what is the value of a European call option?

$S=44$

$K=40$

$r=0.08/\text{year}$

$T=67$ days

$\sigma=0.30/\text{year}$

A dividend of \$1.10 will be paid 39 days from today, i.e., in 0.10685 of a year.

$$S^* = 44 - 1.10e^{-0.08(0.10685)} = 44 - 1.091 = 42.909$$

$$d_1 = \frac{\ln(S^*/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(42.909/40) + [0.08 + 1/2(0.09)]0.184}{0.3\sqrt{0.184}}$$

$$= \frac{\ln(1.0727) + 0.023}{(0.3)(0.42844)} = \frac{0.07018 + 0.023}{0.1285} = 0.7247$$

$$N(d_1) = 0.7658$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.725 - 0.30\sqrt{0.184} = 0.7247 - 0.1285 = 0.5962$$

$$N(d_2) = 0.7245$$

$$Ke^{-rT} = 40e^{-(0.08)(0.18356)} = 40 \times 0.9854 = 39.4169$$

Therefore,

$$C = S^* N(d_1) - Ke^{-rT} N(d_2) = (42.909 \times 0.7658) - (39.4169 \times 0.7245) = 4.299$$

The FinCAD function aaBSdcf computes the same result, as shown in Figure 18.3.

AaBSdcf		
Underlying price	44	
Exercise price	40	
expiry date	15-May-98	
value (settlement) date	9-Mar-98	
Volatility	0.3	
rate curve	t_48_1	
accrual method	1	actual/ 365 (fixed)
option type	1	Call
Statistic	1	fair value
Discounting method	2	continuously compounded rate
Dividend payment table	t_14	

t_48_1
holding cost curve

Maturity date	yield to maturity
9-Apr-98	0.08
9-May-98	0.08
9-Jun-98	0.08

t_14
Dividend payment table

all the dividend dates from the value date to the expiry date	Dividend amount
17-Apr-98	1.1

fair value **4.298630947**

Figure 18.3 Using the FinCAD function aaBSdcf to compute option valuation when the underlying asset pays a discrete dividend.

the expiration date. Here, it is assumed that the ex-dividend day and the payment day are the same, that the dividends are known with certainty, and that the stock price will decline by the dividend amount on the ex-day. Thus, to value a European call option on a dividend-paying stock, merely use S^* in place of S in the standard BSOPM.[10]

18.5.2 Variant Two: The Black–Scholes Option Pricing Model on Securities Paying a Continuous Dividend Stream

Although individual stocks do not pay dividends continuously, a continuous dividend flow model can be used to approximate the dividend stream paid by a well-diversified stock portfolio. The approximation model is particularly inviting if the option has a long time to expiration. However, if the call has a short time to expiration, this model is less adequate. This is because many stocks in the portfolio will trade ex-dividend on the same date during the option's remaining life. At other dates, however, no stock in the portfolio will trade ex-dividend.

This continuous dividend flow model is also valid when one is valuing European calls on futures contracts, on foreign exchange, or on a pure discount debt instrument such as a T-bill. Interest accrues on the T-bill daily, a process that is analogous to a continuous dividend stream.

To use the continuous dividend model, define

$$S^* = Se^{-dT}$$

where d is the annual *constant* dividend yield and T is the time to option expiration, in years. Thus, if a stock index is currently at $S = 156$, the dividend yield is 4%, and the call has nine months to expiration, then

$$S^* = 156e^{-(0.04)(0.75)} = 151.3895$$

Therefore, to value a European call on a stock that pays a constant dividend yield, one would simply substitute the value of S^* for S in the BSOPM.

The FinancialCAD function aaBSG computes the value of a European option on an asset paying a continuous dividend (called a "holding cost" on the FinCAD example).

18.5.3 European Puts with Known Dividends

To value a European put on a stock that will pay known dividends, you can first use the method explained in Section 18.5.1 to value a European call on a dividend-paying stock. Then, you could use the version of put–call parity that applies to European options on stocks that pay known dividends.[11] Alternatively, define S^* as the current stock price minus the present value of all dividends between today and expiration. Then, substitute S^* into the BSOPM put pricing model, Equation (18.8).

The FinCAD functions described earlier typically have an "option switch" button that toggles between a call and a put. For example, aaBS has such a switch on line 8, and aaBINdcf has it on line 10.

18.6 The Relation Between the Binomial and the Black–Scholes Option Pricing Models

Under certain conditions, the BOPM will converge to the BSOPM. Let the number of intervals in the BOPM become infinitely large. The length of each time interval in the BOPM thus shrinks to approach zero. Then, we must choose specific values of u, d, and q (the probability of an "uptick"). These values will result in the stock following a geometric Brownian motion. Thus, let the following define u, d, and q:

$$u = e^{\sigma\sqrt{T/n}} - 1$$

$$d = e^{-\sigma\sqrt{T/n}} - 1$$

$$q = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\frac{T}{n}}$$

where T is the time to expiration and n is the number of subintervals in T. The parameters μ and σ define the expected return on the stock and the standard deviation of the stock's return distribution, respectively. If u, d, and q take on these specific values, it can be shown that as $n \Rightarrow \infty$, the stock price will be lognormally distributed, and the BSOPM will result.

EXAMPLE 18.4 Suppose $T=4$ months $=0.3333$ year, and $n=4$. If the stock's standard deviation of annual returns is 0.60 and the expected return is 0.20 per year, then choosing the following will make the BOPM and BSOPM mutually consistent:

$$u = e^{\sigma\sqrt{T/n}} - 1 = e^{(0.60)\sqrt{0.0833}} - 1 = e^{0.1732051} - 1 = 0.18911$$

$$d = e^{-\sigma\sqrt{T/n}} - 1 = e^{(-0.60)\sqrt{0.0833}} - 1 = e^{-0.1732051} - 1 = -0.159035$$

$$q = \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\frac{T}{n}} = \frac{1}{2} + \frac{1}{2}\left(\frac{0.20}{0.60}\right)\sqrt{0.08333} = 0.5481125$$

If u, d, and q are defined in this way, the stock price will become lognormally distributed as $n \Rightarrow \infty$.[12]

Sometimes, you may wish to use the BOPM, when you are uncertain of the choice of u and d. If you have estimates of the required return on the stock and the volatility of the stock's returns, these formulas can be used to obtain reasonable estimates for u and d. The value of q in the BOPM is only important when you are not assuming a risk-neutral world.

18.7 The Nettlesome Task of Estimating a Security's Volatility, σ

18.7.1 Historical Volatility

The estimate of a call option value made by using the BSOPM is only as good as the estimate of σ that is used in the formula. No one knows what the standard deviation of stock returns will be during the life of the option. Thus, it must be estimated. One way to estimate future volatility is to use historical price data To do this, follow this easy procedure.

1. Decide on the length of the interval for which you will use historical prices. You may wish to select daily, weekly, or monthly prices. Also record any ex-dividend days and any ex–stock distribution days during the estimation period.

2. How many price observations should you gather? There is a trade-off between obtaining too many and too few observations. Assuming no major changes in the company, an estimate of the standard deviation will be more reliable (efficient) as more observations are used. However, as you move further back in time to obtain price data, there will be an increasing likelihood that the risk of the company's stock was different from what it is today. Thus, perhaps a good compromise is to use about 60–200 daily price observations, 40–60 weekly price observations, or 30–50 monthly price observations.

3. Compute the continuously compounded rate of return for each interval. That is, compute daily, weekly, or monthly log price relatives. To do this, find

$$r_{t2} = \ln\left[\frac{S_{t2}}{S_{t1}}\right]$$

or, if there was an ex-dividend day during the interval,

$$r_{t2} = \ln\left[\frac{S_{t2} + \text{div}}{S_{t1}}\right]$$

where div is the dividend amount. You must also adjust these returns for any stock dividends or stock splits.[13]

4. Compute the average of the time series of the n returns, $\bar{r}$:

$$\bar{r} = \left(\frac{1}{n}\right)\sum_{t=1}^{n} r_t$$

where n is the number of log price relatives you have.

5. Estimate the variance of the stock's returns:

$$\text{var}(r) = \left(\frac{1}{n-1}\right)\sum_{t=1}^{n}(r_t - \bar{r})^2 \tag{18.9}$$

Finally, use Equations (18.1) or (18.2) to obtain an estimate of the variance or standard deviation of the stock's returns for any desired interval of time. For example, if you used daily log price relatives, you can estimate the variance of the stock's annual returns by multiplying the variance of daily returns by 365.

EXAMPLE 18.5 Estimating the Variance of a Stock's Monthly Returns

Month	Stock Price	Div	$\frac{S_{t2}+\text{div}}{S_{t1}}$	$\ln\left[\frac{S_{t2}+\text{div}}{S_{t1}}\right]$	$\ln\left[\frac{S_{t2}+\text{div}}{S_{t1}}\right]-\bar{r}$	$\left(\ln\left[\frac{S_{t2}+\text{div}}{S_{t1}}\right]-\bar{r}\right)^2$
0	40.00					
1	42.00	0	1.050000	0.048790	0.041735	0.001742
2	41.125	0	0.979166	−0.021050	−0.028100	0.000790
3	42.375	0	1.030395	0.029942	0.022887	0.000524
4	39.75	0	0.938053	−0.063940	−0.071000	0.005042
5	40	0	1.006289	0.006269	−0.000790	0.000001
6	40	1	1.025000	0.024692	0.017638	0.000311
7	41	0	1.025000	0.024692	0.017638	0.000311
				$\bar{r} = 0.007055$		$\sum_{\text{month}=1}^{7} = 0.008720$

$$\sigma^2 = \left(\frac{1}{6}\right)\sum_{t=1}^{7}(r_t - \bar{r})^2 = 0.001453$$

Thus, the estimated variance of the stock's monthly returns is 0.001453. The variance of the stock's annual returns is 12 times as great: 0.017436. As stated earlier in practice, you would use between 30 and 50 observations to obtain a good estimate of the stock's monthly volatility.

Historical volatility estimates for CBOE-traded options are available at the board's website (www.cboe.com/MktData/HistoricalVolatility.asp).

18.7.2 Improving on the Estimate from Historical Data

The series of closing prices is not the only kind of data that can be used to estimate stock return volatility. For example, you can utilize the information in the stock's ending price, as well as its high, low, and beginning price that occur during the interval.[14]

You can also estimate the historical variances of the stocks of other companies that are in similar businesses and have similar capital structures to compare the stock's historical volatility to that of these firms. If there are significant differences, you may wish to adjust your estimated volatility to be closer to the other firms. Similarly, firm-specific information such as capital structure, liquidity, and fixed versus variable operating costs might aid in estimating a stock's variance. If any of these last data are used, you should generally make only small adjustments to your initial estimated volatility obtained from historical data.

18.7.3 Implied Volatility (IV)

Up until now, you have placed an estimated variance into the BSOPM to obtain an estimated call value. However, it is also possible to take the market price of the call as given and "back out" a variance (or standard deviation) that is consistent with that option price. This variance is called an option's implied variance (implied standard deviation), or, more commonly, implied volatility, VOL, or IV.

The primary advantage of an implied standard deviation over a historical volatility is that an implied standard deviation represents an ex-ante market assessment of risk. For this reason, option-implied forecasts of return volatility have frequently been regarded as superior to estimates based on historical data. Generally, a computer is used to perform a trial-and-error routine to search for the unique σ that provides a model price equal to the observed price.[15]

FinCAD has functions that allow implied volatility to be calculated for options of many different types. For example, the FinCAD function aaBS_iv can be used to compute the implied volatility of a European call or put option. Figure 18.4 illustrates how the implied volatility for Example 18.6 is computed.

EXAMPLE 18.6 On February 19, 1998, the S&P 500 Index closed at 1028.28. The March call options with a strike price of 1030 had a closing price of 20. The short-term interest rate was 5.25%. The option had 28 days until expiration.

Solution A standard deviation of 16.23% is consistent with this option price, as shown in Figure 18.4.

aaBS_iv		
Underlying price	1028.28	
Exercise price	1030	
Expiry date	20-Mar-98	
Value (settlement) date	19-Feb-98	
Price	20	
Risk free interest rate	0.0525	
Option type	1	call
Discounting method	1	annually compounded rate
accrual method	1	Actual/365 (fixed)
implied volatility	**0.162318349**	

Figure 18.4 The FinCAD function aaBS_iv can be used to determine the implied volatility of a call option.

If you use yesterday's option price data to obtain an IV to use in valuing options today, an obvious question is: Which option price should be used? There have been several techniques suggested for making the best use of information contained in a set of implied volatilities. Despite the widespread use of implied volatility, there is no generally accepted method to estimate implied volatility.[16]

18.7.4 Using Implied Volatility Estimates

Finding the implied volatilities of options is useful for determining which option is the most undervalued or the most overvalued. Relative to the other options on the same stock, the option with the lowest IV is most undervalued. This statement assumes that the assumptions behind the BSOPM are valid. However, there could be good reasons for one option to have a higher or lower IV than another option. For example, the stock's volatility might be expected to change over time, causing calls with longer times to expiration and calls with shorter times to expiration to have different IVs. In addition, there could be a perceived probability of a jump (a discontinuity) in the stock price, which would cause calls with different strikes to differ in their IVs. Ex-dividend dates can cause IVs to differ across options' times to expiration. Finally, refer to Section 18.7.6, which discusses the "volatility smile."

Computing an IV that is drastically different from one you believe ought to be incorporated in an option's price is tantamount to holding the belief that the option is mispriced. Suppose you calculate an IV for a call to be 30% (based on its asked price). However, you believe that the true volatility is 60%. In other words, you believe the call is undervalued. You could just buy the call. On average, if your beliefs are correct, you should earn an above-average rate of return for the risks you bear.

Alternatively, you can try to arbitrage by purchasing the undervalued call and selling the equivalent portfolio of stock and T-bills. The proportions allocated to stock and T-bills in the latter portfolio should reflect the estimated volatility of 60%. The equivalent portfolio must be revised frequently over time, to reflect changes in the required proportions that are needed to replicate the call. If your beliefs about volatility are correct, you should end up with a profit equal to the difference between the value of the call with a volatility of 30% (its actual price), and the value of the call if it were selling with a volatility of 60%.

Merville and Pieptea (1989), analyzed the time series properties of the IVs of 25 stocks. They disclosed at least two interesting findings: changes in IV are correlated across stocks; and there appear to be one or more forces that pull IVs back to their long-term average values (this is called **mean reversion**). These findings imply that the IV of one stock provides information about the IV of another stock. In addition, if a recent IV is considerably different from its long-run average IV (perhaps calculated over the past year), then a "better" estimated volatility might be a weighted average of the recent IV and its long-term mean.

Many traders use the mean reverting property of implied volatility to improve their trading performance. For example, suppose a bullish trader believes that a certain stock price is about to rise. If the current IV is below its long-term average value, the trader will buy calls. As such, he is effectively buying the underlying asset and buying volatility. Note that the trader might be wrong and the price of the underlying asset might *not* rise substantially, but he can still profit. If the IV of the option does increase, the trader will benefit, because all else equal, the call price will increase as its IV increases.

If the IV is above its long-term average value, option prices are "high," so the trader will write puts, rather than buy calls. Here, he is selling volatility. If the puts' IV declines to its long-run average value, the put prices will decline, and the put seller will profit even if the stock price does not rise.

18.7.5 Market Volatility Index (VIX)

The volatility implied in option prices is important financial information. Beginning in 1993, the Chicago Board Option Exchange began computing and disseminating a real-time, market-wide implied volatility index. Thus, during the trading day, investors can monitor the market's assessment of expected stock market risk by looking up the symbol (VIX). By its construction, the Market Volatility Index, is a 30-day forward-looking measure of the dispersion of expected returns of the Standard & Poor's 100 Index (OEX).

Whaley (1993) describes in detail how the VIX is constructed as well and discusses how the VIX can be used. Briefly, however, the VIX is calculated as follows. Each minute during the trading day, the implied volatility from eight OEX options is calculated. These eight options are divided into two groups, one group representing options with the nearest expiration date and the other representing options with the second-nearest expiration date. Both groups of options consist of the two calls and two puts with strike prices immediately above and below the current OEX level. After the implied volatilities have been calculated, they are weighted such that the VIX represents an implied volatility of an at-the-money OEX option with 30 calendar days to expiration. Data on the VIX can be found at the exchange's website (www.cboe.com/MktData/vix.asp).

18.7.6 The Volatility Smile and the Term Structure of Volatility

If the Black–Scholes model is correct, then all options with the same underlying asset should have the same implied volatility, regardless of their strike prices or times to expiration. If options systematically differ in their implied volatilities, then either the BSOPM is wrong (perhaps because of flawed assumptions behind the model) or the market is mispricing the options.

Given actual trade prices, there is often a structure to volatilities implied by the BSOPM. This volatility structure is generally divided into two parts: a **volatility smile** (also known as the "skew") and a **term structure of volatility**. The volatility smile describes the way implied volatility varies across strike prices for a given expiration date. The "term structure of volatility" is the relationship among implied volatility across expiration dates for a given strike price.[17]

As an example of the volatility smile, consider the data in Table 18.1. where the option premium column gives average bid–ask prices for June 2000 S&P 500 Index call and put options at the close of trading on May 16, 2000. Such data are available each day at the CBOE website (www.cboe.com). Then, taking the May 16, 2000, closing S&P 500 Index level of 1466.04, a riskless interest rate of 5.75%, an estimated dividend yield of 1.5%, and T=0.08493 year (31 days to expiration), we can use the FinCAD function, aaBSG_iv, to generate implied volatilities.

The columns of Table 18.1 present the strike price, the call price, the call's implied volatility, and the call price, computed with a standard deviation of 0.22 (which is approximately the implied volatility of the around-the-money options). Then the put price, the put's implied volatility, and the put price computed with a standard deviation of 0.22 are presented in the last three columns.

From Table 18.1, one can see that implied volatility is not constant across strike prices. In addition, comparing columns 2 and 4 shows that if you use the constant at-the-money implied volatility of 0.22 to price *all* the calls, there are systematic pricing errors. That is, the in-the-money call options appear to be overpriced (their actual prices are above their theoretical prices) and the out-of-the-money call options appear to be underpriced (their actual prices are below their theoretical prices).

The implied volatilities of put prices are also not the same for all strike prices. But interestingly, the *out*-of-the money puts have higher implied volatilities than the in-the-money puts. For calls, the reverse is generally true. That is, for strikes up to 1600, the out-of-the-money calls have

TABLE 18.1 Implied Volatility Across Strike Prices

Strike Price	Call Price	Call IV	Theoretical Call Price with $\sigma = 0.22$	Put Price	Put IV	Theoretical Put Price with $\sigma = 0.22$
1225				1.5625	0.3360	0.054
1250				2.34375	0.3283	0.153
1275				2.625	0.3021	0.390
1300				3.8125	0.2920	0.904
1325				5.75	0.2855	1.919
1350	129.5	0.2834	124.335	8.25	0.2762	3.753
1375	107.625	0.2689	102.510	11.375	0.2641	6.809
1400	86.625	0.2534	82.353	14.875	0.2463	11.533
1425	67.625	0.2423	64.288	20.0625	0.2315	18.350
1450	50.6875	0.2324	48.648	29	0.2285	27.592
1475	35.625	0.2201	35.612	38.625	0.2152	39.437
1500	22.5	0.2036	25.178	50.875	0.2016	53.885
1525	14	0.1982	17.173	66.75	0.1923	70.762
1550	7.875	0.1912	11.291	85.375	0.1826	89.761
1575	4.375	0.1896	7.154	106.875	0.1788	110.505
1600	2.28125	0.1881	4.367	129.5	0.1668	132.600
1625	1.21875	0.1897	2.569	153.375	0.1512	155.684
1650	0.625	0.1912	1.457			

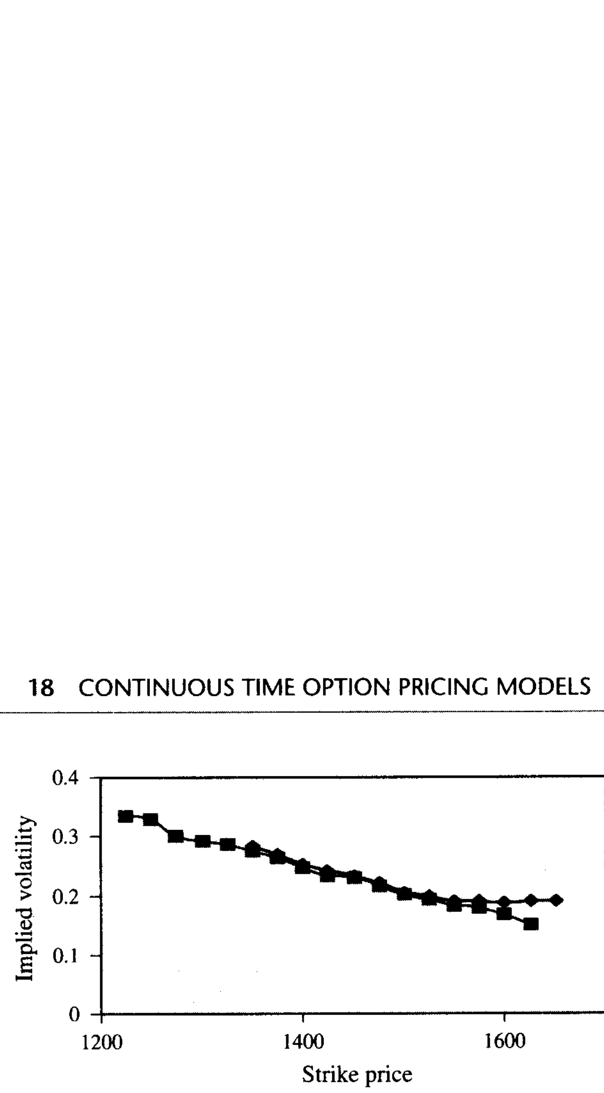


Figure 18.5 Volatility smiles: diamonds, call IV; squares, put IV.

lower implied volatilities than in-the-money calls. But the "smile" appears for the calls when strikes exceed 1600.

Figure 18.5 graphs the volatility smile data presented in Table 18.1. Actually, only the calls exhibit a slight smile (perhaps a smirk?). Puts show only a decline in their implied volatilities as strike prices rise.

Traders should account for any relationship between σ and K when they use the BSOPM to value options. Otherwise, if they use just one constant volatility to value all options, they will conclude that options are systematically over-or undervalued, depending on how far they are in or out of the money. One way to account for the volatility smile uses the following quadratic regression equation:

$$\text{IV} = \alpha + \beta_1 \ln\left(\frac{S}{K}\right) + \beta_2 \ln\left(\frac{S}{K}\right)^2 + \varepsilon \tag{18.10}$$

This model states that the implied volatility in option prices is a quadratic function (U-shaped, if β_2 is positive) of S/K, which measures the extent to which the option is in or out of the money. Using the call data in Table 18.1 to estimate Equation (18.10) results in the following information:

$$\alpha = 2.169318 \ (t = 6.639)$$
$$\beta_1 = -4.47227 \ (t = -6.727)$$
$$\beta_2 = 2.522705 \ (t = 7.484)$$
$$R^2 = 0.988$$

These coefficients are unlikely to be constant over time. Therefore, traders should use the most recently available price data to estimate the model. This method is a simpler approach than estimating a stochastic volatility model.[18]

Volatility smiles may occur if volatility itself is stochastic. In other words, an assumption behind the BSOPM is that the volatility of the underlying asset is constant, right through expiration. But a more accurate depiction of security markets is that the volatility of returns changes from day to day. Many individuals in the market spend considerable time trying to accurately predict future volatility. Volatility may trend up and down, and it may randomly fluctuate around the trend. Occasionally, jumps (discontinuities) in the time series of prices occur. Volatility that changes over time, and/or jumps, can create volatility smiles.

18.8 Generalizing the Black–Scholes Option Pricing Model

It is important to understand that many option pricing models are related. For example, you will see in Chapter 20 that many exotic options are simply parts of the BSOPM. The BSOPM itself, however, can be generalized to encapsulate several important pricing models. All that is needed is the addition of a cost-of-carry term, b. Then, one can use this generalized model to price European options on non-dividend-paying stocks (Black and Scholes, 1973), options on stocks that pay a continuous dividend (Merton, 1973), currency options (Garman and Kohlhagen, 1983), and options on futures (Black, 1976). The formulas for the generalized options are as follows:

$$C_{\text{gen}} = Se^{(b-r)T}N(d_1) - Ke^{-rT}N(d_2) \tag{18.11}$$

and

$$P_{\text{gen}} = Ke^{-rT}N(-d_2) - Se^{(b-r)T}N(-d_1) \tag{18.12}$$

where, S, K, r, T, $N(d)$, σ, ln, and e are as defined in Section 18.2. However, in the generalized BSOPM, we have

$$d_1 = \frac{\ln(S/K) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \tag{18.13}$$

$$d_2 = \frac{\ln(S/K) + (b - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{18.14}$$

and

b = cost-of-carry rate of holding the underlying asset

By altering the 'b' term in Equations (18.11) through (18.14), four option pricing models emerge.

Setting	**Yields This European Option Pricing Model**
$b=r$	Stock option model (i.e., the BSOPM)
$b=r-\delta$	Stock option model with continuous dividends, δ
$b=r-r_f$	Currency option model where r_f is the foreign risk-free rate
$b=0$	Futures option model

18.9 Options on Futures Contracts

Because we have spent considerable time examining options and futures, it is natural to spend a bit of time learning about options on futures contracts. A futures option is an option on a futures contract. A call futures option gives the owner the right but not the obligation to assume a long position in a futures contract. The buyer of a futures call pays a premium to the seller (writer). The writer of a futures call has the obligation to deliver a futures contract should the call owner decide to exercise it. The strike price is the futures price at which the long position is assumed. Upon exercising a futures call, the positions are marked to market. Thus, whereas the exerciser of an "ordinary" call must pay $\$K$ to acquire the asset, the exerciser of a futures call is actually paid the intrinsic value of the option. That is, the futures contract is immediately marked to market, and the exerciser is paid $\$(F-K)$ to go long a futures contract, where F is the futures settle price on the exercise day, and K is the striking price.

Futures options trade on many different exchanges, with underlying assets consisting of virtually every successful futures contract. The *Wall Street Journal* presents their prices in tables that are based on the underlying commodity. For example, see Figure 14.8. Different conventions exist for interpreting futures options price data. Therefore, before you ever trade futures options, be sure to learn how to read the data!

Many futures options have expiration dates in months that are not delivery months for the futures contracts that underlie them. For example, there are January, February, and March S&P 500 futures options (in fact, you can trade S&P 500 futures options with any of 12 expiration months). But S&P 500 futures contracts expire quarterly: in March, June, September, and December. The same situation exists for all foreign exchange futures that trade on the IMM. What this means is that January, February, and March futures options have March futures underlying them. April, May, and June futures options require the delivery of June futures contracts, and so on. An investor who exercises a *February put* on a Japanese yen futures contract, will go short one *March* yen futures contract at the strike price.

The last trading day of futures options differs from contract to contract, and it can differ for a given contract, depending on whether the option expires in the futures' delivery month or in a non-delivery month. Always contact the exchange on which a futures option (or futures contract) trades for current contract specifications, as these can change at any time.

18.9.1 Valuing European Futures Options

Black (1976) derived a formula for computing the value of a European call option on a forward or futures contract. Black made all the assumptions that were behind the BSOPM, including constant interest rates. This last assumption is critical because if interest rates are known, a futures price is equivalent to a forward price.[19]

From Equations (18.11) through (18.14), by setting $b=0$, and using F to denote the underlying asset, Black's model for call options is generally written as follows:

$$C = e^{-rT}\left[FN(d_1) - KN(d_2)\right] \tag{18.15}$$

and for puts, it is

$$P = e^{-rT}\left[KN(-d_2) - FN(-d_1)\right] \tag{18.16}$$

The delta of a European futures call is

$$\Delta_c = e^{-rT}N(d_1); \quad 0 \le \Delta_c < 1$$

and the delta of a European futures put is

$$\Delta_p = -e^{-rT}\left[1 - N(d_1)\right]; \quad -1 < \Delta_p \le 0$$

Note the similarity to the BSOPM. The only difference is that Fe^{-rT} is substituted for S. Intuitively, this makes sense because Fe^{-rT} is the present value of F, in continuous time. Thus, the Black model combines the BSOPM with the cost-of-carry futures pricing model.

Observe Equations (18.15) and (18.16). Suppose that a futures call is very deep in the money so that $N(d_1)=N(d_2)=1.0$. Then Equation (18.15) yields a theoretical call value of

$$C = e^{-rT}[F - K]$$

This is the present value of its intrinsic value, which is less than its intrinsic value of $F-K$. This means that a deep in-the-money European futures call can sell for less than its intrinsic value (unless r or T is zero), regardless of whether there are payouts such as dividends on the underlying asset. Similarly, if a futures put is very deep in the money, then

$$P = e^{-rT}\,[K - F]$$

which is the present value of $K-F$. Thus, we can conclude that like calls, European futures puts can also sell for less than their intrinsic value.

Finally, it is logical to expect the Black model to underprice exchange-traded futures options because it is a European futures options pricing model. However, all futures options that trade in the United States are American futures options. We will see next that both American futures calls and puts can exercised early if they are sufficiently in the money. Therefore, the prices of American futures options will almost always have an early exercise premium above that of their European equivalents.

18.9.2 Valuing American Futures Options

All the futures options that trade in American markets are American-style contracts. Accordingly, the owners of these options can exercise them at any time before expiration. Because owners might find it optimal to exercise early, American futures options cannot be worth less than otherwise identical European futures options. Also, an American futures option cannot sell for less than its intrinsic value.

There will always exist a critical futures price, F^*, for which early exercise will be optimal. For American futures calls, it is the futures price at which the call sells for its intrinsic value. If $F>F^*$, then the futures call owner will find it optimal to exercise early. The owner of an American futures call might find it optimal to exercise early even when there are no dividends or carry returns. Note that this is different from a call on a spot good, which will be exercised early only just before an ex-dividend date. For American futures puts, F^* is the price at which the put sells for $K-F$ (its intrinsic value); if F is below F^*, it will be rational to exercise the American futures put early.

We see intuitively, that deep in-the-money futures options are exercised early because when an investor exercises them, he receives a cash inflow, since the futures position is marked to market. By not exercising early, the investor is missing out on interest income on the futures option's intrinsic value.

Because deep in-the-money futures puts and calls might be exercised early, American futures options will almost always be worth more than European futures options.[20]

18.9.3 Put–Call Parity for Options on Futures Contracts

For European futures options, the put–call parity proposition is:

$$C - P = (F - K)(1 + r)^{-T} = PV(F - K) \tag{18.17}$$

Note that Equation (18.17) combines the standard cost-of-carry pricing model $[(F = S(1+r)^T = S + \text{carry costs})]$ with the standard put–call parity theorem.[21]

For American futures options, the put – call parity proposition is:[22]

$$F(1 + r)^{-T} - K \le C - P \le F - K(1 + r)^{-T} \tag{18.18a}$$

or

$$PV(F) - K \le C - P \le F - PV(K) \tag{18.18b}$$

18.9.4 Strategies That Use Options on Futures Contracts

Any option strategy covered in Chapter 15 can be replicated with futures options and futures contracts. Note, however, that when one is going long or short futures, there is no initial cash flow; in contrast, when one is buying and selling stock, there are initial cash outflows and inflows, respectively.

For example, a bullish stock market investor can use S&P 500 futures options to speculate on that belief. He can buy futures calls. Alternatively, he can reduce his initial outlay by buying a vertical futures call spread. These strategies offer one significant advantage over simply going long the March S&P 500 futures contract: there is a limited loss if the bullish investor is wrong.

A crude oil producer might purchase put options on crude oil futures contracts as insurance against the risk that the spot price of crude oil will decline below the price at which the firm will no longer earn a sufficient profit.

For another example, consider an investor who believes that the $/¥ exchange rate will be stable in the near future. The investor can sell a strangle on the Japanese yen by selling a futures call and selling a futures put.

Next, suppose a manager of a portfolio of Treasury bonds expects interest rates to remain stable, or perhaps rise somewhat. He can use T-bond futures options to his advantage. He might consider writing futures calls as a way to increase income. Note that this is not exactly a covered call position because instead of being long T-bond futures contracts, the manager is long spot bonds.

Consider an equity portfolio manager who would commit funds to the market if the S&P 500 fell another 10 points. He might consider writing out-of-the money naked futures puts as a way of increasing income. Here, the sale of the futures puts serves as a substitute for placing limit buy orders.

Thus, we see that futures options offer market participants opportunities to speculate on their beliefs, and to hedge against adverse price changes.

18.10 American Call Options

If there are no dividends, the basic BSOPM values American calls as well as European calls. The reason for this is that absent dividends, we know that rational investors will not exercise American call options early. If the underlying asset does pay dividends, however, there are several methods of estimating the value of an American call. These include the following:

1. The BOPM
2. Pseudo-American call valuation model
3. Roll–Geske–Whaley compound option model
4. Numerical methods and simulation

We have already covered the BOPM (Chapter 17). The BOPM provides estimates as accurate as those of methods 3 and 4 at no greater effort or computer time. Refer to Jarrow and Turnbull (2000, pp. 257–258), and Hull (2000, Appendix 11B) to learn about the Roll–Geske–Whaley model. Chapter 16 of Hull (2000) and Part Six of Wilmott (1998) cover numerical methods and simulation. Here, we will discuss only the pseudo-American approximation model.

18.10.1 Pseudo-American Call Model

Recall that an American call will be exercised before expiration on a day before the underlying stock trades ex-dividend, if at all. Suppose there are two ex-dividend days before expiration, $t1$ and $t2$:

$$0 \qquad t1 \qquad\qquad t2 \qquad T$$

As with the discrete dividend pricing model for European options (see Section 18.5.1), we assume that we can decompose today's stock value into two components: a riskless component that equals the present value of the dividends at times $t1$ and $t2$, and a risky component that consists of the present value of all dividends after time T.

Suppose you knew that the call was going to be exercised immediately before time $t1$, that is, at time $t1-\varepsilon$ (ε represents an instant). Then the life of the option is only from time 0 to time $t1$. Although the call exerciser pays \$$K$ for the stock at time $t1-\varepsilon$, she also will immediately receive the dividend, div1. Moreover, by owning the stock at time $t1-\varepsilon$, she will also be entitled to receive the present value of the second dividend, $\text{div2}(1+r)^{-(t2-t1)}$. Thus, define the following values:

$$S^* = S - PV(\text{divs}) = S - \text{div}(1+r)^{-t1} - \text{div2}(1+r)^{-t2}$$

$$K^* = K - \text{div1} - \text{div2}(1+r)^{-(t2-t1)}$$

$$T^* = t1$$

To find the value of this option, which we will refer to as C_S (S stands for "short"), substitute the S^* for S, K^* for K, and T^* for T into the BSOPM.

Suppose instead that you knew that the call was going to be exercised just before the second dividend, that is, at time $t2-\varepsilon$. Now, the exerciser pays \$$K$ for the stock, and she will immediately receive the dividend, div2. Define:

$$K^{**} = K - \text{div2}$$

$$T^{**} = t2$$

To find the value of this option, denoted as C_M (M stands for "medium"), substitute S^* for S, K^{**} for K, and T^{**} for T, into the BSOPM.

Finally, suppose you knew that the option would not be exercised early. Then the value of the option equals the value of a European call on a dividend-paying stock expiring at time T. As explained in Section 18.5.1., such a call can be valued by substituting S^* into the basic BSOPM. Refer to this option as C_L (L stands for "long").

EXAMPLE 18.7 Suppose the call in the example of Section 18.5.1. was American. Because there is only one ex-dividend date, we need consider only the values of a short-lived option C_S and a long-lived option C_L. The example in Section 18.5.1 provides the value of C_L: \$4.298. The value for C_S is computed as follows:

$S = 44$

$K = 40$

$r = 0.08$/year

$T = 67$ days = time until expiration day

$\sigma = 0.3$/year

A dividend of \$1.10 will be paid 39 days from today.

$S^* = 44 - 1.10e^{-0.08(0.10685)} = 44 - 1.091 = 42.909$

$K^* = 40 - 1.1 = 38.90$

$T^* = 39/365 = 0.10685$ year

$$d_1 = \frac{\ln(S^*/K^*) + (r + \sigma^2/2)T^*}{\sigma\sqrt{T^*}} = \frac{\ln(42.909/38.9) + [0.08 + 1/2(0.09)]0.10685}{0.30\sqrt{0.10685}}$$

$$= \frac{\ln(1.10306) + 0.0134}{(0.30)(0.3269)} = \frac{0.098087 + 0.013356}{0.098064} = 1.13644$$

$N(d_1) = 0.8721$

$d_2 = d_1 - \sigma\sqrt{T^*} = 1.13644 - 0.30\sqrt{0.10685} = 1.13644 - 0.098064 = 1.0384$

$N(d_2) = 0.85045$

$K^* e^{-rT^*} = 38.90e^{-(0.08)(0.10685)} = 38.90 \times 0.9915 = 38.5689$

Therefore,

$$C_s = S^* N(d_1) - K^* e^{-rT^*} N(d_2) = (42.909 \times 0.8721) - (38.5689 \times 0.85045) = 4.6205.$$

You are encouraged to use the FinCAD function aaBS to check this answer. In this example, because the pseudo-American call model in Equation (18.19) states that the call will be the greater of C_S and C_L, the value of the American call is max(4.298, 4.6205), or \$4.6205.

According to the pseudo-American call valuation model, the value of an American call on a dividend-paying stock is the greatest of C_S, C_M, and C_L:[23]

$$C = \max(C_S,\ C_{M,}\ C_L) \tag{18.19}$$

FinancialCAD offers two programs to compute the value of American options: aaBIN2 for continuous dividends and aaBINdcf for discrete dividends. If aaBINdcf is used to value the pseudo-American call option example in Section 18.10.1, a theoretical value of 4.709 is found. As predicted, the pseudo-American model underpriced this option. Figure 18.6 shows the results from aaBINdcf.

18.10.2 American Puts

Because they might be exercised early, American puts present the same valuation difficulties as American calls on dividend-paying stocks. Actually, the problem is worse, because American puts may be exercised anytime. In contrast, American calls will be exercised only just before an ex-dividend date.

Geske and Shastri (1985) discuss the factors that increase the likelihood that American puts will be exercised early. They find that American puts will most likely be exercised early immediately after an ex-dividend date. This is logical because a put holder would typically expect the stock price to decline on the ex-dividend date, so he will be inclined to wait until after the next ex-dividend date before exercising. Furthermore, some investors own the underlying stock plus protective puts. These investors will frequently wish to receive any dividends that are forthcoming.

aaBINdcf		
underlying price	44	
exercise price	40	
expiry date	19-Sep-2000	
value (settlement) date	14-Jul-2000	
volatility	0.3	
rate - annual compounding	0.08	
accrual method - riskless rate	1	Actual/365 (fixed)
dividend payment table	t_14	
option type	1	Call
number of iterations	25	
statistic	1	fair value

t_14
dividend payment table

all the dividend dates from the value date to the expiry date	dividend amount
22-Aug-2000	1.1

fair value 4.708767055

Figure 18.6 The FinancialCAD function aaBINdcf is used to value American calls when the underlying asset pays a discrete dividend.

If so, they may be inclined to wait until after the ex-date before exercising their puts. Lower interest rates and transactions costs will also discourage early exercise.

In the absence of dividends, there will always exist some critical stock price below which the holder of an American put will exercise. That critical stock price is the price at which an American put sells for exactly its intrinsic value, $K-S$. In many cases, the critical stock price is not too much below the strike price. For example, for an option with the parameters $K=\$20$, $r=10\%$, $\sigma=0.4$/year, and $T=3$ months, the critical stock price is \$14.75. At that stock price, an otherwise equivalent European put is valued at less than its intrinsic value, while the American put is worth its intrinsic value of 5.25. Figure 18.7 illustrates the pricing of an American and a European put given the foregoing parameter values.

The BSOPM works reasonably well for short term, out-of-the-money puts because the probability of early exercise is low. However, when the probability of early exercise is non trivial, several other methods exist to value American puts. These include the following:

1. The BOPM
2. Numerical methods
3. Approximation techniques

The BOPM was applied to the valuation of American puts in Section 17.4. The FinCAD functions aaBIN2 and aaBINdcf can be used to value American puts; these functions employ the binomial option pricing model. Numerical methods lie beyond the scope of this book. However, several approximation methods to value American puts exist. These methods include a technique introduced by Johnson (1983), the compound option approach of Geske and Johnson (1984), and quadratic approximations derived by MacMillan (1986) and Barone-Adesi and Whaley (1987).

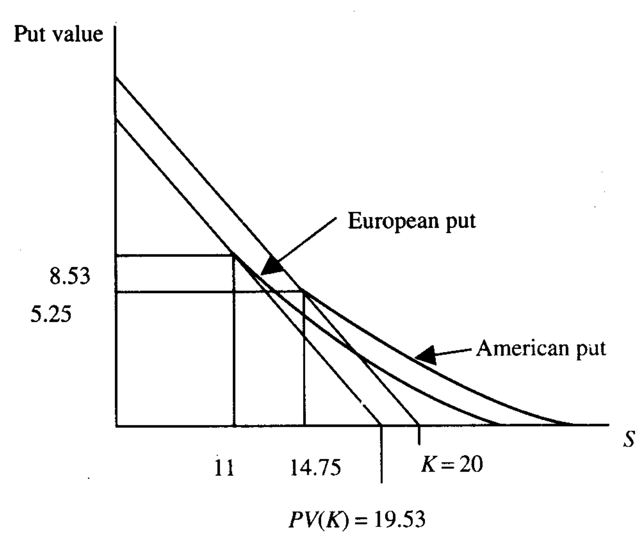


Figure 18.7 Relative pricing of an American and a European put when $K=\$20$, $T=3$, months, $r=10\%$, and $\sigma=40\%$/year. At prices below the critical stock price, $S^*=\$14.75$, the American put is worth its intrinsic value. At and below S^*, the American put will be exercised early.

18.11 Summary

This chapter provides a summary of some of the continuous time option pricing models that have been developed. The BSOPM is the easiest to use because it is a formula that can be solved by hand. However, the BSOPM is developed under critical assumptions. Figlewski (1989a, 1989b), who studies the impact of real markets (in which the cited assumptions do not hold) on option pricing, has concluded that in practice, the BSOPM will provide only guidelines for option values. Figlewski finds that option prices can lie within rather wide bounds without permitting any arbitrage profits from trading the underlying asset, bonds and the options. In particular, he cites the importance of accurately predicting the underlying asset's volatility and the problems caused by indivisibilities, brokerage fees, and bid–asked spreads. These factors make call replication (hence arbitrage) difficult. In other words, the BSOPM ignores many real considerations that also affect actual option prices. As such, the model should be used only to compare prices of different options, not to establish definitive option values.

Even when the BSOPM is used as an approximation, remember that the model is developed under a specifically assumed stochastic process for the underlying asset, and thus it may not provide accurate values for American options that are likely to be exercised early. In contrast, the more flexible BOPM can always be used to value an option on an asset that follows any type of stochastic pricing process, and also can value American options. Other option pricing models have been derived to value options when the underlying asset follows specific pricing processes. Other techniques have been developed to aid us in valuing American options. However, these techniques almost always require the use of a computer.

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Notes

[1]See Black and Scholes (1973). Many papers contributed to the development of the Black–Scholes option pricing model. See Smith (1976, footnote 3 and section 3), for a summary of several of these models.

[2]Ramaswamy and Sundaresan (1985) incorporate stochastic (i.e., randomly changing) interest rates into a stock index futures option pricing model.

[3]If a stock's returns are lognormally distributed, the log of the stock's relative prices, that is, $\ln(S_t/S_{t-1})$, is normally distributed. Note that the log of the stock's relative prices equals the stock's continuously compounded returns.

[4]It is also frequently written as $C = SN(d_1) - K(1+r)^{-T}N(d_2)$. This version assumes discrete discounting to permit computation of the present value of the strike price.

[5]See the appendix to this chapter for a table of the cumulative standard normal distribution function.

[6]See Chapter 10.

[7]However, the current price of the stock is a function of its expected return. Thus, if new information were suddenly revealed that caused investors to expect a stock to double in price (i.e., its expected return is 100%), we would expect that S would quickly be revalued to reflect its equilibrium required rate of return. The latter is a function of the stock's risk.

[8]Several numerical approximation formulas exist to compute the values of $N(d_1)$ and $N(d_2)$. For example, a handy approximation, accurate to two decimal places if $0<d<2.20$ is given by:

$$N(d) \approx 0.5 + \frac{d(4.4-d)}{10}$$

For $d_1 = 0.064745$, $N(d_1)$ is approximately 0.528068. The same formula works for d_2, when $0<d_2<2.20$. Thus, to compute $N(d_2)$, where $d_2 = -0.045255$, one must take advantage of the symmetry of the normal distribution. That is, $N(-d_2) = 1 - N(d_2)$. Therefore, to compute $N(-0.045255)$, we would write

$$\begin{aligned} N(-d_2) &= 1 - N(d_2) \\ &\approx 1 - \left(0.5 + \frac{0.045255(4.4 - 0.045255)}{10}\right) \\ &\approx 0.48029 \end{aligned}$$

Chriss (1997) provides additional numerical approximations accurate to four, six, and more decimal places.

[9]A similar intuitive interpretation of $N(d_1)$ does not exist. However, the whole term, $S_0\,N(d_1)$, can be interpreted as the expected value (using risk-neutral probabilities) of the stock price conditional on the stock price exceeding the exercise price times the probability that the option will expire in the money.

[10]An implicit assumption for this model is that σ is the volatility of only the portion of today's stock price that consists of the present value of dividends after expiration. The model essentially says that S consists of two components: a riskless part, which is the present value of dividends between today and expiration, and a risky part, which is S^*. The volatility of S^* is described by σ.

[11]The version is $C - P = S - PV(\text{divs}) - Ke^{-rT}$.

[12]These definitions of u, d and q are given in Cox and Rubinstein (1985, p. 200), but they are not unique. Using a slightly different technique, Jarrow and Turnbull (2000, p. 135) define the appropriate values of u, d, and q to be

$$\begin{aligned} q &= \frac{1}{2} \\ u &= e^{\left[\left(r-(\sigma^2/2)\right)(T/n)+\sigma\sqrt{T/n}\right]} \\ d &= e^{\left[\left(r-(\sigma^2/2)\right)(T/n)-\sigma\sqrt{T/n}\right]} \end{aligned}$$

In the example just presented in the text, if $r = 12\%$ per year, then Jarrow and Turnbull's formulas would result in $u = 1.183173$ and $d = -0.836775$. Then $S_u = Se^u$ and $S_d = Se^d$ using the Jarrow and Turnbull notation.

[13]For example, if S_{t2} is an ex-stock split price of 40.625, S_{t1}, the last price before the ex-day, is 80, and the split size is 2, then the return is:

$$\ln\left[\frac{(2)(40.625)}{80}\right] = 0.0155042$$

[14]For a discussion of these extreme value methods, see Yang and Zhang (2000). Cho and Frees (1988) develop a variance estimator that employs every observed trade during some intraday time interval. Their estimator makes use of bid and ask prices and the length of time between price changes.

[15]However, some closed-form estimators exist. For example, see Brenner and Subrahmanyam (1988) and Corrado and Miller (1996a).

[16]For a discussion of various methods, see Corrado and Miller (1996b).

[17]Plotting both effects simultaneously in a three-dimensional picture results in a **volatility surface**.

[18]Shimko (1993) offers a similar approach to account for the volatility smile.

18.15 You may wonder whether there is a difference between the implied volatility of a European call on a dividend-paying stock and the implied volatility of an American call on the same dividend-paying stock. FinCAD may provide a clue to this question. Use aaBSdcf_iv and aaBINdcf_iv to compute the implied volatility for the situation in which the stock sells for $60/share and it will trade ex-dividend one month from "today"; the dividend amount is $1/share. The call option itself has a strike price of 55, it expires two months from today, and the call premium is 7. The riskless interest rate is 6% for all maturities on the yield curve. Explain your results.

Partial answer: The implied volatility for the European call is 47.225%. You should work with FinCAD until you come up with this answer. Then proceed to compute the implied volatility for the American call.

18.16 Look in a recent *Wall Street Journal*. Choose an option that satisfies the following:

a. There must be a put and a call with the same underlying asset and the same strike price.

b. At least 100 puts and 100 calls must have traded.

c. The time to expiration is greater than two weeks and less than three months.

Find the ticker symbol for your underlying asset (www.cboe.com/tools/symbols/ can help you here). Find the historical volatility for your underlying asset (go to www.cboe.com/tools/historical/ and click on the most recent textfile with the data). Then use the FinCAD function aaBS to compute the theoretical fair value for your put and for your call.

18.17 A deep in-the-money European put can sell for less than its intrinsic value. An otherwise equivalent American put must sell for at least its intrinsic value. Use the FinCAD function aaBS to find the price at which a European put sells for less than its intrinsic value. Start with an at-the-money put, with the underlying asset and the strike price both equal to 50. The option expires two months from today. The volatility of the underlying asset is 35%. The riskless interest rate is 6%. You should find, using aaBS, that the fair value of the European put is 2.602122697. Continue to work with the inputs (lines 2–11) for aaBS until you come up with this solution. ***Then***, start reducing the price of the underlying asset by $1 at a time, until the fair value of the put is less than intrinsic value. Print the page at which this happens.

Partial Answer

Price of Underlying Asset	Intrinsic Value	Fair Value of Put
49	1	3.074543875
48	2	3.603770217
47	3	4.190380375

Note how the put's theoretical value is approaching its intrinsic value. Continue with this process until you find the price of the underlying asset that produces a put value that is less than its intrinsic value. ***Finally***, use aaBIN to find the theoretical value of an American put, taking as the underlying asset price the one that produced a fair value for the European put that was less than its intrinsic value.

18.18 Lets check how closely put–call parity holds for index LEAPS (long-term equity appreciation security). Go to http://quote.cboe.com/QuoteTable.htm. Click on "all options and LEAPS for this underlying (complete file)". The underlying index is SPX (the S&P 500 Index option). Scroll down the resulting file (which may take a minute or two to get) until you see "XX DEC" options, where XX are the last two digits of next year (e.g., in November 1999, XX = 00). Choose the at-the-money strike and record the midpoint of the bid–ask spread for the put and the call. For example, on November 19, 1999, the S&P was at 1422, so the following data were observed.

Calls

	Bid	Ask
00 Dec 1425 SXG LQ-E	151 1/4	155 1/4

Puts

	Bid	Ask
00 Dec 1425 SXG XQ-E	95 5/8	99 5/8

Use the basic put–call parity proposition, $C-P=S-PV(K)$ to determine how closely put–call parity holds when actual prices are used. Then, use the FinCAD function aaBSG to estimate the theoretical fair values for your put and your call. Note that the "holding cost" is the dividend yield on the index, which you should estimate from recent information (*Barron's* presents this information each week). Estimate the riskless interest rate from data in a recent *Wall Street Journal*. The volatility of the S&P 500 is typically about 22%. Options expire on the third riday of the month. Print the aaBSG output files.

18.19 The Black–Scholes option pricing model will often underprice the true value of American options. Explain why.

18.20 Suppose that Mr. Bull buys a call option on a December, 2001, S&P 500 futures contract. The call costs him $1750, has a strike price of 1375, and expires on the third Friday of December. The December futures price on that day is 1318.

a. At what price does Mr. Bull have the right to go long a December futures?

b. Does Mr. Bull have to hold the futures call option until expiration? Explain.

c. Suppose that before the third Friday of December, the December S&P 500 futures price rises to 1395.45 and the futures call premium rises to $2750. Mr. Bull decides to exercise the futures call option. Calculate his cash inflow.

d. On subsequent days, his long futures position is marked to market daily. Calculate Mr. Bull's cash flow if the S&P 500 futures price rockets to 1400.

18.21 Suppose the December corn futures price is 333.75. Calculate the value of an option on this futures contract, assuming $K=350$, the riskless interest rate is 7.50%/year, there are 79 days to expiration, and $\sigma=20\%$/year.

APPENDIX Notes on Continuous Compounding and Stock Return Distributions

A18.1 CONTINUOUS COMPOUNDING

Suppose you have $1.00 today, and a bank pays interest of 6%, compounded annually. Then one year from today, you will have $1.06, and two years hence, you will have $1(1.06)(1.06)=\$1.1236$.

If the bank pays interest of 6%, compounded monthly, then it is paying 1/2% interest each month. Thus, after one month, you will have $1.005. After two months, you will have $1(1.005)(1.005)=\$1.010025$. After 12 months, you will have $1(1.005)^{12}=\$1.06168$. Thus, a 6% interest rate, compounded monthly, provides you with a 6.168% annual rate of return. After 24 months, your original dollar will be worth $1.12716.

Define the following terms:

F = future value

P = present value

r = interest rate

m = number of times interest is compounded per year (e.g., $m = 12$ if interest is compounded monthly).
t = number of years

The general formula for determining the future value of a present amount, if interest is compounded is:

$$F = P\left(1 + \frac{r}{m}\right)^{mt}$$

In this example, if $m = 12$ and $t = 2$, we would have

$$F = 1\left(1 + \frac{0.06}{12}\right)^{(12)(2)} = \$1.12716$$

Interest can be compounded at smaller intervals of time. For example, interest can be compounded daily, or by the second. As m gets larger, the more frequently interest is compounded. As m approaches infinity (∞), it can be shown that

$$F = Pe^{rt} \tag{A18.1}$$

Like π, which equals 3.1415927 ..., e is a specific number, namely, 2.7182818. Thus, if you deposit \$1.00 into a bank that pays 6% interest, compounded continuously, then after one year, you will have

$$F = 1e^{(0.06)(1)} = \$1.0618$$

and after two years, you will have

$$F = 1e^{(0.06)(2)} = \$1.1275$$

If your calculator has an "e^x" button, then verify that $e^{0.06} = 1.0618$ and that $e^{0.12} = 1.1275$. If your calculator has a "y^x" button verify the foregoing equation by using $y = 2.7182818$.

If you are told that you are earning c% on your money, compounded continuously, then you can compute the effective annual rate of return, a%, by using this relationship:

$$e^c - 1 = a$$

For example, $e^{0.06} - 1 = 0.0618$. In other words, 6% compounded continuously is equivalent to 6.18% compounded annually. See example A18.1 for an application.

Given an annual rate of a%, you can reverse the procedure to find the equivalent continuously compounded rate, c%:

$$e^c - 1 = a$$
$$e^c = 1 + a$$
$$\ln(e^c) = \ln(1 + a)$$
$$c = \ln(1 + a)$$

EXAMPLE A18.1 **What is the effective annual rate that is equivalent to earning 15% compounded continuously?**

Answer $e^{0.15} - 1 = 16.1834\%$.

Thus if you earn a rate of return of $a\%$ per period, finding the natural logarithm of $(1+a)$ will provide you with the equivalent continuously compounded rate of return.

If a stock rises from \$14/share to \$18/share, its rate of return is $(18-14)/14 = 0.2857143 = 28.57143\%$. Its continuously compounded rate of return is $\ln(S_{t2}/S_{t1}) = \ln(18/14) = \ln(1.2857143) = 0.2513144$. This 25.13144% continuously compounded rate of return is equivalent to an "uncompounded" rate of return of 28.57143%.

It is stated later (Section A18.2.2) that if a stock follows geometric Brownian motion, the distribution of its returns will be lognormally distributed. If returns are lognormally distributed, the log of the stock's relative prices are normally distributed. The log of relative prices are equal to the stock's continuously compounded returns.

A18.2 STOCK RETURNS DISTRIBUTIONS

A18.2.1 Geometric Brownian Motion

One assumption used in deriving the BSOPM is that the stock price randomly wanders through time following a price path called geometric Brownian motion. Although much of the background material necessary to understand geometric Brownian motion is mathematically quite difficult, the implications of the stock price following this particular *stochastic process* is important for a discussion of the BSOPM. Thus, what follows is a discussion that is aimed at helping users of option models decide whether the value of the underlying stock or asset moves randomly according to geometric Brownian motion or some other stochastic process.

The stochastic process that the underlying asset must follow in order to derive the BSOPM is called geometric Brownian motion, and it is defined by:[1]

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \Delta z \tag{A18.2}$$

where

$\Delta S/S = (S_{t+\Delta t} - S_t)/S_t$ = rate of return on the stock

μ = expected (constant) return on the stock per unit of time

Δt = a unit of time

σ = the (constant) standard deviation of the stock's return during the unit of time

z = a normally distributed random variable with a mean of zero and a variance of t.

The random variable z, is called a Wiener process. Over small intervals of time, changes in z, Δz, are normally distributed random variables, with $E(\Delta z) = 0$, and $\text{var}(\Delta z) = \Delta t$. The covariance of any two Δz is zero; in other words, $\text{cov}(\Delta z_{t2}, \Delta z_{t1}) = 0$.

Note that Δz is just a normally distributed random variable, with a mean of zero and a variance of Δt. Suppose Δt is one day. We draw a value of Δz out of a probability distribution that has a mean of zero and a variance of one. If, instead, we are interested in Δz over one week, and Δt is one day, then Δz is drawn out of a probability distribution having a mean of zero and a variance of 7 (because there are seven days in a week). The mean is always zero and the variance is proportional to time. In addition, any two realizations of Δz are independent.

EXAMPLE A18.2 Suppose Δt is one day. A stock has an expected return of $\mu = 0.0005$ per day.[2] The standard deviation of the stock's daily return distribution is 0.0261725.[3] The return generating process is such that each day the return consists of a nonstochastic component, 0.05%, and a random component. The latter equals the stock's daily standard deviation times the realization of Δz, which is drawn from a normal probability distribution with a mean of zero and a variance of one.

Table A18.1 depicts one particular realization of the stochastic process driving this stock's price over a 60-day period. The first column is the day number. The second column is the realization of Δz, a random number drawn from a normal distribution with a mean of zero and a variance of one. The third column illustrates how the stock price would move if there were no stochastic component to its return. That is, each day, the stock's return is 0.05%. The fourth column gives the stochastic component, $\sigma\Delta z$, where $\sigma = 0.0261725$/day. The daily return, R, is $\mu\Delta t + \sigma\Delta z$. Column five lists each day's stock price, where $S(T) = S(T-1)[1 + R]$.

TABLE A18.1 An Example of a Stock Price Path Over a 60-day Period Driven by Geometric Brownian Motion

Day	Δz	Nonstochastic Trend Price $S(T)=S(T-1)[1.0005]$	Stochastic Component $(0.0261725)\Delta z$	$S(T)=S(T-1)[1.0+R]$ $R=\mu\Delta t+\sigma\Delta z$ $\Delta t=1$ Day
0		1.000000		1.000000
1	−2.48007	1.000500	−0.064910	0.935590
2	−0.87537	1.001000	−0.022911	0.914623
3	−0.80587	1.001501	−0.021092	0.895789
4	−1.03927	1.002002	−0.027200	0.871871
5	0.10523	1.002503	0.002754	0.874709
6	0.66993	1.003004	0.017534	0.890483
7	−0.21137	1.003505	−0.005532	0.886002
8	2.19733	1.004007	0.057510	0.937398
9	−0.82807	1.004509	−0.021673	0.917551
10	0.58783	1.005011	0.015385	0.932126
11	−1.25487	1.005514	−0.032843	0.901978
12	−0.26827	1.006017	−0.007021	0.896096
13	1.28023	1.006520	0.033507	0.926569
14	0.56773	1.007023	0.014859	0.940800
15	−0.03447	1.007526	−0.000902	0.940422
16	1.29413	1.008030	0.033871	0.972745
17	0.06143	1.008534	0.001608	0.974795
18	0.79553	1.009038	0.020821	0.995578
19	1.66593	1.009543	0.043601	1.039485
20	−0.44497	1.010048	−0.011646	1.027899
21	−0.03137	1.010553	−0.000821	1.027569
22	0.36873	1.011058	0.009650	1.037999

TABLE A18.1 Continued

Day	Δz	Nonstochastic Trend Price $S(T)=S(T-1)[1.0005]$	Stochastic Component $(0.0261725)\Delta z$	$S(T)=S(T-1)[1.0+R]$ $R=\mu\Delta t+\sigma\Delta z$ $\Delta t=1$ Day
23	−0.20397	1.011563	−0.005338	1.032977
24	−0.13357	1.012069	−0.003496	1.029882
25	0.34653	1.012575	0.009069	1.039737
26	0.20593	1.013082	0.005390	1.045861
27	−0.04727	1.013588	−0.001237	1.045090
28	−0.64737	1.014095	−0.016943	1.027905
29	−0.41207	1.014602	−0.010785	1.017333
30	−0.06837	1.015109	−0.001790	1.016021
31	−0.20927	1.015617	−0.005477	1.010964
32	−0.55077	1.016125	−0.014415	0.996897
33	−0.38087	1.016633	−0.009968	0.987458
34	0.27863	1.017141	0.007292	0.995152
35	−0.44457	1.017650	−0.011636	0.984071
36	−1.07717	1.018158	−0.028192	0.956819
37	0.17163	1.018667	0.004492	0.961596
38	0.50863	1.019177	0.013312	0.974877
39	0.78913	1.019686	0.020653	0.995499
40	−0.49757	1.020196	−0.013023	0.983033
41	1.32373	1.020706	0.034645	1.017582
42	0.83613	1.021217	0.021884	1.040359
43	−1.82237	1.021727	−0.047696	0.991258
44	−0.38177	1.022238	−0.009992	0.981849
45	−1.17227	1.022749	−0.030681	0.952215
46	1.26993	1.023261	0.033237	0.984341
47	0.35793	1.023772	0.009368	0.994054
48	1.84673	1.024284	0.048333	1.042597
49	−1.35187	1.024796	−0.035382	1.006229
50	−0.76187	1.025309	−0.019940	0.986668
51	−0.47267	1.025821	−0.012371	0.974955
52	−0.22147	1.026334	−0.005797	0.969791
53	−1.20347	1.026847	−0.031498	0.939730
54	0.41413	1.027361	0.010839	0.950385
55	−0.78127	1.027875	−0.020448	0.931427
56	1.59143	1.028388	0.041652	0.970688
57	0.44953	1.028903	0.011765	0.982594
58	1.48893	1.029417	0.038969	1.021376
59	−0.72487	1.029932	−0.018972	1.002509
60	−0.50337	1.030447	−0.013175	0.989803

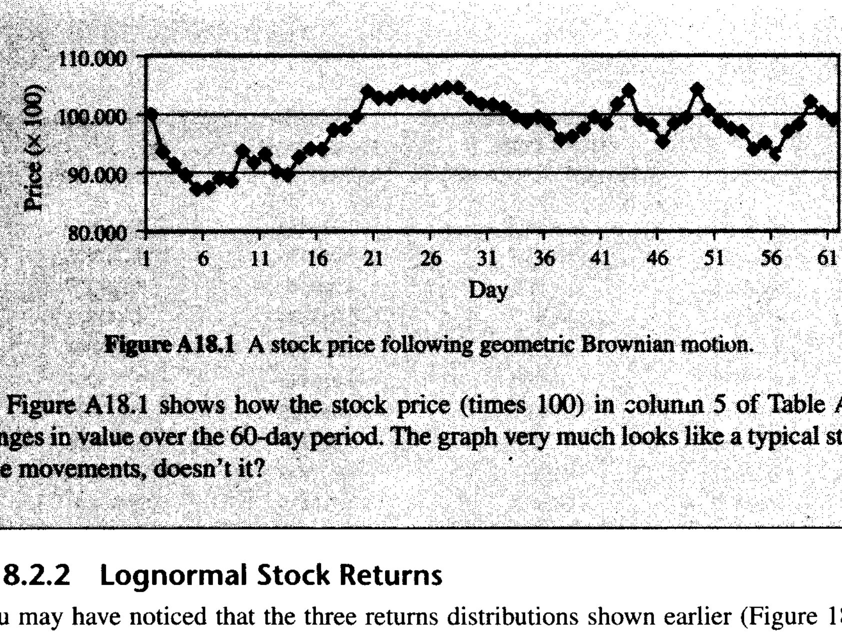

Figure A18.1 A stock price following geometric Brownian motion.

Figure A18.1 shows how the stock price (times 100) in column 5 of Table A18.1 changes in value over the 60-day period. The graph very much looks like a typical stock's price movements, doesn't it?

A18.2.2 Lognormal Stock Returns

You may have noticed that the three returns distributions shown earlier (Figure 18.1a–c) are not normally distributed. In fact, one result of the stock following a geometric Brownian motion is that returns are "lognormally" distributed. Specifying that returns are lognormally distributed is equivalent to saying that the natural logarithm of the relative prices, $\ln(S_{t2}/S_{t1})$, are normally distributed.

Lognormal returns are realistic for two reasons. First, if returns are lognormally distributed, then the lowest possible return in any period is -100%. In contrast, if returns are normally distributed, there is some probability that returns will be less than -100%. The difference between normally distributed returns and lognormally distributed returns is illustrated in Figure A18.2.

If a security's return during a period is -100%, the terminal stock price at the end of the period, S_{t2}, is zero. Then, $\ln(S_{t2}/S_{t1}) = \ln(1+R) = \ln(0) = -\infty$, which is the "left tail" of any normal distribution. But the left tail of a lognormal distribution is anchored at 0 when R, the return, is -100%.

Second, lognormal returns distributions are "positively skewed." Positive skewness is characterized by the extended right tail in Figures 18.1 and A18.2. This is realistic because while the lowest return in any period is -100%, the highest return will likely be in excess of 100% when measured over a year. Thus, a realistic depiction of a stock's returns distribution would have a minimum return of -100% and a maximum return well beyond $+100\%$. The longer the time interval under consideration, the more valid the latter statement becomes. Therefore, annual returns will be more positively skewed than monthly returns, and monthly returns will be more skewed than daily returns. Returns distributions will not be symmetric. They will be skewed to the right.

There is one last point to consider. It was stated that the natural logarithm of a stock's relative prices will be normally distributed if the stock follows the pricing process we are describing. The natural logarithm of relative prices, $\ln(S_{t2}/S_{t1}) = \ln(1+R)$, is a continuously compounded return. In other words, if a stock price follows a geometric Brownian motion, its continuously compounded returns are normally distributed. Its returns measured over any longer interval of time are lognormally distributed.

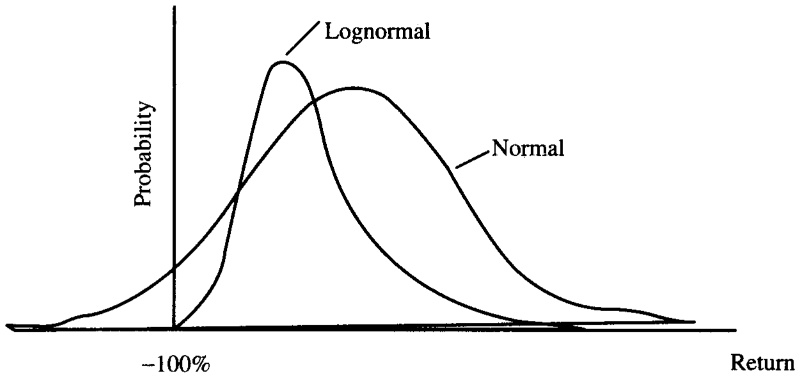

Figure A18.2 Lognormal and normal probability distributions.

To summarize, then, three important concepts of this section are:

1. An important assumption of the BSOPM is that the stock price follows a geometric Brownian motion.
2. As a result of this assumed stock price behavior, returns over a given interval of time are lognormally distributed and continuously compounded returns are normally distributed.
3. If the stock's price follows a geometric Brownian motion, its returns variance is proportional to time and the standard deviation of its returns is proportional to the square root of time (see Section 18.1.3).

Assuming that geometric Brownian motion governs the movements of the underlying stock price is important. In particular, in some instances the geometric Brownian motion is restrictive. For example, in the geometric Brownian motion process, the stock price changes by just a little bit every instant. While this is an adequate description for most stocks at most times, it does not fit the stock's situation in all cases. For instance, suppose a stock is a target of an unfriendly takeover that is being resisted. Suppose the stock price is midway between the offer price and the stock price before the tender offer. If the acquisition is successful, it is likely that the stock price will jump, instantaneously, by $10/share. However, if the target defends itself, it is likely that the stock price will decline by $5/share. Under these circumstances, the stock price would be well described as usually following a continuous diffusion process. However, at random times, the stock price "jumps" a random amount up or down. If there is a nontrivial probability of a jump in the price of the underlying asset, the BSOPM may not be adequate when valuing options on that asset.[4]

Another problem is that in the geometric Brownian motion model, the variance of the stock's returns is assumed to be constant regardless of the stock's price. However, there is considerable evidence that stock prices are more volatile at lower prices than at higher prices. In addition, considerable research has concluded that returns variances themselves change randomly over time; that is, they are stochastic. This seems reasonable, because both the nature of the firm and its environment randomly change over time. The variance of a stock's returns distribution will likely be greater on a day that an earnings report is released, a dividend announcement made, or a government statistic on the economy released, than on other days. Several models have been developed to price options under conditions of stochastic volatility. These papers also cite considerable research that supports, both theoretically and empirically, the idea that stock returns variances randomly change over time.[5]

A18.3 Cumulative Probabilities for the Standard Normal Distribution

TABLE A18.2a Cumulative Probabilities for the Standard Normal Distribution: Negative Values of z

	Second Digit of z									
z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
−3.5	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Example: $N(-0.22)=0.4129$

TABLE A18.2b Cumulative Probability for the Standard Normal Distribution: Positive Values of z

	Second Digit of z									
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998

Example: $N(0.64) = 0.7389$

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Notes

[1]Actually, the stochastic process exists in continuous time. Thus, Δt is an "instant", μ is the instantaneous expected rate of return, σ is the standard deviation during one instant of time, and the returns process is:

$$\frac{dS}{S} = \mu dt + \sigma dz$$

[2]A return of 0.0005/day is 0.05%/day. If this were compounded over 365 days, the expected annual return would be $(1.0005)^{365} - 1 = 0.20016$, or 20.016% per year.

[3]If the standard deviation of the stock's daily returns is 0.0261725, the variance of the stock's daily returns is 0.000685, the variance of the stock's annual returns is $(365)(0.000685) = 0.250025$, and the standard deviation of the stock's annual returns is $(0.250025)^{0.5} = (0.0261725)(365)^{0.5} = 0.500025$.

[4]For additional detailed discussions on jump process option pricing models, see Cox and Ross (1976), Merton (1976a, 1976b), Cox, Ross and Rubinstein (1979), Jarrow and Rudd (1983, Chapter 12), Ball and Torous (1985), and Wilmott (1998, Chapter 26).

[5]For additional detailed discussions on stochastic volatility models, see Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), Finucane (1989), Heston (1993), and Wilmott (1998, Chapter 23).

CHAPTER 19

Using Options for Risk Management

Volume in option trading continues to remain robust. In this chapter, we focus on the details of how options provide price insurance. That is, how can a risk manager protect an underlying portfolio from adverse price changes using options? As you learned in earlier chapters, options can be used to limit downside risk while still allowing upside participation. These results can be provided by either the fiduciary call strategy (purchase calls and debt instruments) or the protective put strategy (the owner of an asset buys puts). Also, options can be used to limit upside risk while reaping any benefits from declines in the price of an underlying asset. This can be achieved by buying calls to provide insurance against a short position.

Also in this chapter, we will present some necessary technical details concerning how option values change as the factors that influence option values (S, K, r, T, σ) change. Risk managers must be aware of these details to use options effectively. Although option contracts are indispensable tools for risk management, market participants use options for other reasons. These include the following.

- *Options can generate additional cash flow.* The sale of covered calls provides additional cash flow. Of course, the writer of a covered call also hopes that the underlying asset price does not rise much above the strike price. Writing naked puts is a revenue-providing strategy that is used as a substitute for placing limit orders to buy an asset.
- *Options can be used to exploit tax-related situations.* Writing a covered call as a substitute for the outright sale of the asset might defer a capital gain, or stretch a short-term gain into a long-term gain. Many other tax-driven strategies exist, but users should always obtain an opinion from tax accountants or tax attorneys before attempting to use options to reduce taxes.
- *Options provide leverage.* Because the purchase of a call is equivalent to buying the underlying asset and borrowing, the leverage provided by options may exceed that available to many market participants who purchase only the underlying asset. The initial premium of an option is generally only a small fraction of the cost of buying the underlying asset.
- *Options can circumvent short selling difficulties.* If an asset cannot be easily sold short, the purchase of a put may be the most efficient method for generating profits from a price decline in the underlying asset.

An important question every risk manager faces is whether to buy options to insure against adverse price moves or to use forwards, futures, or swaps to hedge against price risks. Unfortunately, there is no easy answer to this question. However, here are some important considerations.

When options are used to buy insurance, there is an initial cash outflow, the option premium. This is often a negative factor in the decision to use options to manage risk. By contrast, forwards, futures, and swaps can often be used with no initial cash outflow. Recall that forwards, futures, and swaps can often lock in a price. That is, these contracts can sometimes reduce price uncertainty to zero. However, this might be a negative factor in the decision to use these derivatives to manage risk. Using forwards, futures, and swaps instead of options means that in 50% of the cases, the risk manager will be likely to regret having hedged a spot position. This occurs when an unhedged position would have benefited from the subsequent price change. When used as insurance, options hedge only downside risk. The insurer will capture the upside, less the cost of insurance (the option premium). These differences are illustrated in Figure 19.1 for the situation in which a firm has a long position in the underlying asset and faces the risk that prices will fall.[1]

In Figure 19.1a, the sale of futures or forwards hedges the long position in the underlying asset. The result is the horizontal line with a zero change in profit, regardless of price changes. Figure 19.1b illustrates how the purchase of a protective put insures against downside risk, but allows for profitable participation should the price of the underlying asset rise. Thus, risk managers must understand the trade-off of having to *pay* for insurance vs the alternative of hedging, which has zero initial cost. Given this trade-off, the risk manager must next consider his beliefs about the direction of prices and his tolerance for taking risks of adverse price movements.

Suppose the policy of a firm is to have a continuous hedging strategy.[2] This firm's risk manager expects prices to move in a way that would actually benefit the firm if it was unhedged. This risk manager may wish to use options, so that the firm will benefit from the beneficial subsequent price change. Under the situation in Figure 19.1, she might want to buy the protective put. This will cost an initial put premium, but if she is correct and prices do rise (by an amount large enough to offset the cost of the put), the firm will be better off than it would have been, had it hedged by selling forwards or futures.

Furthermore, the risk manager is not bound to just one strategy. It is possible to initially buy the put and, if prices rise, later sell the put (at a loss), and sell futures or forwards to lock in the value of the spot position at a higher price.

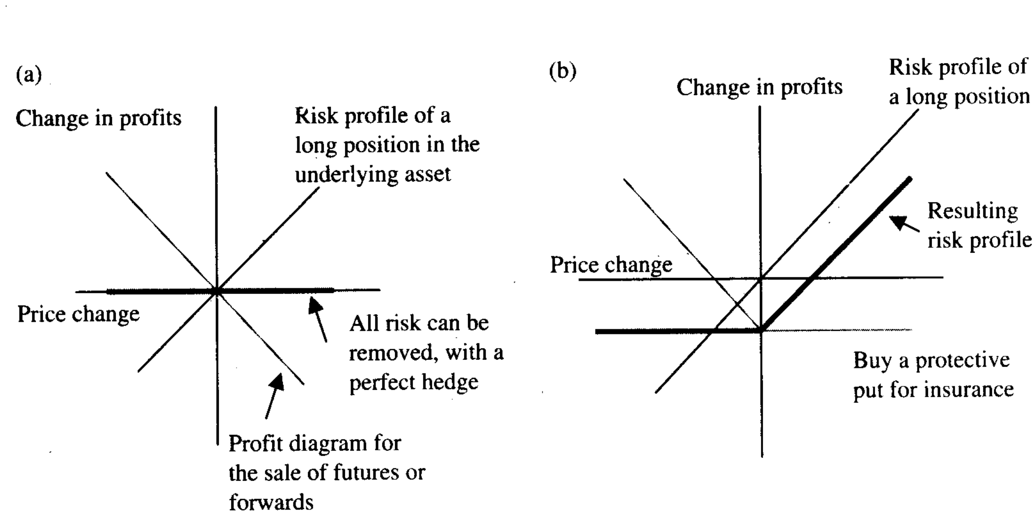

Figure 19.1 Profit diagrams illustrate the difference between (a) hedging and (b) insurance.

Next we discuss "the Greeks," which refer to the analysis of how an option's value changes if there is a change in one of the factors that determines its value (σ, S, r, T, or K). These sensitivities are determined by taking the partial derivative of the Black–Scholes option pricing model (BSOPM) with respect to one of these determinants of value, and each result has traditionally been assigned a Greek letter. Understanding these measures is extremely important for everyone who uses options, particularly those who use them in risk management programs.

19.1 The Greeks

A risk manager cannot make a well-informed decision to use options without knowing how option values change as the factors that influence those values change. This knowledge is necessary for the risk manager to understand how the value of a portfolio consisting of the underlying asset and options will change during the life of the hedge.

The theoretical call value determined by the BSOPM can be partially differentiated with respect to each of its five parameters, K, T, r, σ, and S.[3] The results are formulas that predict how much a call option value will change if only *one* input parameter changes by a small amount, all else equal. That is, no other input parameter values are allowed to change. Economists call such an analysis "comparative statics," or sensitivities. Because each of the sensitivities is commonly known by a Greek letter, the BSOPM comparative statics are also known as "the Greeks." Each of the Greeks has a "sign." A positive sign means that the option value will increase when the factor increases, all else constant. A negative sign means that the option value will decrease when that factor increases, all else equal. To begin, let us look at the Greeks for call options.

19.1.1 The Greeks for Black–Scholes Calls (a.k.a. "No-Name," Theta, Rho, Vega, Delta, and Gamma)

The formulas for the Greeks for Black–Scholes calls are as follows.[4]

Partial Derivative Notation	Greek Letter	Brief Interpretation	Formula	Sign
$\frac{\partial C}{\partial K}$		By how much will a call price change, given a change in the strike price?	$-e^{-rT}N(d_2)$	< 0
$\frac{\partial C}{\partial T}$	theta[5] (θ)	By how much will a call price change, given a change in the time to expiration?	$Ke^{-rT}\left[\frac{\sigma N'(d_2)}{2\sqrt{T}} + rN(d_2)\right]$	> 0
$\frac{\partial C}{\partial r}$	rho (ρ)	By how much will a call price change, given a change in the riskless interest rate?	$TKe^{-rT}N(d_2)$	> 0
$\frac{\partial C}{\partial \sigma}$	vega (ν)	By how much will a call's value change, given a change	$S\sqrt{T}N'(d_1)$	> 0

		in the volatility of the underlying asset?		
$\frac{\partial C}{\partial S}$	delta (Δ)	Hedge ratio: By how much will a call price change given a change in the value of the underlying asset?	$N(d_1)$	>0, but $0 \le \Delta \le 1$
$\frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}$	gamma (Γ)	By how much will a call's delta change, given a change in the price of the underlying asset?	$\frac{N'(d_1)}{S\sigma\sqrt{T}}$	>0

In these formulas, note that there are terms for $N(d_1)$ and for $N'(d_1)$, where $N'(d)$ is the height of the standard normal density at d (and d can represent any value of d_1 or d_2). It can also be thought of as the incremental change in the area under standard normal distribution at d. It is given by

$$N'(d) = \frac{\partial N(d)}{\partial d} = \frac{e^{-d^2/2}}{\sqrt{2\Pi}}$$

Recall that the normal distribution function sums the area under the normal curve from $-\infty$ to d. The normal density measures the height of the normal curve at d. Thus, if there is a small change from d to d', the distribution function increases by the value of the density at d.

EXAMPLE 19.1 Comparative Statics

Compute the values of each of the foregoing partial derivatives for the following call option:

$S=47$

$K=50$

$T=0.5$ year

$r=10\%$/year

$\sigma=0.40$/year

Then use the BSOPM to calculate the theoretical call value as follows. First, obtain a value for d_1.

$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(47/50) + [(0.10 + (1/2)(0.16)]0.5}{0.4\sqrt{0.5}}$$

$$= \frac{\ln(0.94) + 0.09}{(0.4)(0.7071)} = \frac{-0.061875 + 0.09}{0.28284} = 0.099435$$

Then, using the table of the cumulative standard normal distribution function in the appendix to Chapter 18 (Table A18.2), we can finish the calculation:

$$N(d_1) = 0.539604$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.099435 - 0.40\sqrt{0.5} = 0.099435 - 0.28284 = -0.18341$$

$$N(d_2) = 0.427239$$

$$Ke^{-rT} = 50e^{-0.05} = 50 \times 0.9512 = 47.5615$$

$$C = SN(d_1) - Ke^{-rT}N(d_2) = (47 \times 0.539604) - (47.5615 \times 0.427239) = 5.0413$$

Now use the same formulas to find the sensitivity of the option's value to each of the parameters. First,

$$\frac{\partial C}{\partial K} = -e^{-rT}N(d_2) = -\left[e^{-(0.10)(0.50)}\right](0.4272) = -0.4064$$

That is, if you were to compare two calls that were identical in every way except that their strike prices were \$1 apart (e.g., $K=49$ vs $K=50$), the one with the lower strike should be worth \$0.4064 more than the one with the higher strike price.

$$\frac{\partial C}{\partial T} = Ke^{-rT}\left[\frac{\sigma N'(d_2)}{2\sqrt{T}} + rN(d_2)\right]$$

$$= (50)\left(e^{-(0.1)(0.5)}\right)\left[\frac{(0.4)(0.3923)}{2\sqrt{0.50}} + (0.1)(0.4272)\right] = 7.3093$$

The value of N'(d2) in this partial derivative formula is given by

$$\frac{e^{-d_2^2/2}}{\sqrt{2\Pi}} = \frac{e^{-(-0.1834)^2/2}}{\sqrt{2(3.1416)}} = 0.3923$$

If the call's time to expiration had been one year longer, the call price would have been about \$7.3093 greater. That is, if the call had had 1½ years to expiration instead of ½ year, it would have been worth about \$12.3506. Actually, because the partial derivative evaluates changes in call values for small changes in T, it is more accurate to estimate that the rate of time decay is about \$0.020025/day (7.3093/365), all else equal. Theta is greatest for at-the-money (or just-in-the-money) options that are close to expiration. Any time value rapidly disappears as the expiration date nears.

$$\frac{\partial C}{\partial r} = TKe^{-rT}N(d_2) = (0.50)(50)\left[e^{-(0.1)(0.50)}\right](0.4272) = 10.159$$

This means that if interest rates were to rise from 10% to 110%, the value of the call would rise by \$10.159. Again, however, it is more accurate to interpret the number as meaning that an increase in the riskless interest rate from 10% to 11% would result in an increase in the value of the call equal to \$0.10159, or about 10 cents. This example illustrates that it is *not* critical to have a precise value for the riskless interest rate.

$$\frac{\partial C}{\partial \sigma} = S\sqrt{T}N'(d_1) = (47)\left(\sqrt{0.50}\right)(0.3971) = 13.193$$

The value of $N'(d_1)$ in the foregoing equation is:

$$\frac{e^{-d_1^2/2}}{\sqrt{2\Pi}} = \frac{e^{-(0.0994^2/2)}}{\sqrt{2(3.1416)}} = 0.39697$$

Thus, if σ were to increase by 1.0, from 0.40 to 1.40, the value of the call would rise by about \$13.193. More realistically, if the volatility increased from 0.40 to 0.50, the call's value would be \$1.3193 greater than originally estimated. Also, note that the call price would be lower by \$1.3193 if the volatility fell to 0.30. Because σ is the only parameter that is not observable, it must be estimated. This example illustrates just how critical an accurate estimate of σ must be! It also demonstrates the potential rewards from buying options when implied volatility is low and expected to rise, and from selling options when implied volatility is high and expected to decline.

Next, let us calculate

$$\frac{\partial C}{\partial S} = N(d_1) = 0.5396$$

This means that if the stock price were to rise by one dollar, we would expect the call value to increase by about \$0.54, all else equal. The delta of a call option must be greater than or equal to 0 but less than or equal to 1. Note that for an at-the-money call option, where $d_1 \approx 0$, the delta is about 0.50. We will discuss the importance of delta in detail later in the chapter.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T}} = \frac{0.39697}{47 \times 0.40 \times \sqrt{0.50}} = 0.02986$$

Given the example just discussed, an increase in the stock price of \$1 will increase the call delta by 0.02986. In other words, currently, the delta of the call is 0.5396. If the stock were to rise in price from \$47/share to \$48/share, the delta would rise to about 0.56946. This Greek is known as gamma, and we will also discuss the importance of gamma later in the chapter.

The FinCAD function AsBS will compute the Greeks for our example, as shown in Figure 19.2. Note that FinCAD follows convention by presenting theta as a negative number for the *one-day* decay in the call price. The actual value (from FinCAD) of -0.020025318 is very close to the value we computed ($7.3093/365 = -0.020025$ per day). FinCAD's value for vega of 0.131930711

AaBS		
underlying price	47	
exercise price	50	
expiry date	29-Jan-2001	
value (settlement) date	31-Jul-2000	
Volatility	0.4	
risk free interest rate	0.1	
option type	1	Call
Statistic	1	Fair value
discounting method	2	Continuously compounded rate
accrual method	1	Actual/365 (fixed)

fair value	5.041250976
Delta	0.539603796
Gamma	0.029862089
Theta	-0.020025318
Vega	0.131930711
Rho	0.101600637

Figure 19.2 The FinCAD function AaBS computers the Greeks.

is the amount the call value will change if the volatility of the underlying asset increased from 0.40 to 0.41. Use FinCAD to compute the value of the call if $\sigma=41\%$ and check this result. Finally, FinCAD's value for rho is 0.101600637, which is the amount that the theoretical call value will rise if the interest rate rises from 10% to 11%.

19.1.2 The Greeks for Black–Scholes Option Pricing Model: Puts

The following equations define how the value of a European put changes, given that any one of the five underlying determinants of option values changes, all else equal. Once the Greeks for calls have been obtained, the Greeks for puts can be obtained by differentiating the variable P in the put–call parity formula $P=C-S+Ke^{-rT}$. (The key is to remember that both P and C are *functions* of K, T, r, S, and σ.)

Partial Derivative Notation	Greek Letter	Brief Interpretation	Formula	Sign
$\frac{\partial P}{\partial K}$		By how much will a call price change, given a change in the strike price?	$\frac{\partial C}{\partial K}+e^{-rT}$	>0
$\frac{\partial P}{\partial T}$	theta[5] $(\theta)^5$	By how much will a put price change, given a change in the time to expiration?	$\frac{\partial C}{\partial T}-rKe^{-rT}$	$\lesseqgtr 0$
$\frac{\partial P}{\partial r}$	rho (ρ)	By how much will a put price change, given a change in the riskless interest rate?	$\frac{\partial C}{\partial r}-TKe^{-rT}$	<0

$\frac{\partial P}{\partial \sigma}$	vega (ν)	By how much will a put's value change, given a change in the volatility of the underlying asset?	$\frac{\partial C}{\partial \sigma} = S\sqrt{T}N'(d_1)$	>0
$\frac{\partial C}{\partial S}$	delta (Δ)	Hedge ratio: By how much will a put price change given a change in the value of the underlying asset?	$\frac{\partial C}{\partial S} - 1 = N(d_1) - 1$	<0, but $-1 \le \Delta \le 0$
$\frac{\partial \Delta}{\partial S} = \frac{\partial^2 P}{\partial S^2}$	gamma (Γ)	By how much will a put's delta change, given a change in the price of the underlying asset?	$\frac{N'(d_1)}{S\sigma\sqrt{T}}$	>0

There are some interesting features about the put sensitivities. First, note that the sign of theta for a put is indeterminate. Mathematically, this is because in absolute value, the second term in the equation can be less than or greater than the theta of a call.[6] Second, the influence of volatility is the same for puts as it is for calls. Third, the delta of a put equals the delta of a call minus one. Consequently, the gamma value of a put equals the gamma value for a call.

19.2 The Importance of Delta

Delta is vital in formulating and evaluating option strategies. Therefore, addition to what has been said thus far in this chapter, as well as in Chapters 17 and 18, Section 19.2.1 is devoted entirely to the concept of an option's delta.

19.2.1 What Is Delta?

The delta, Δ, of a call is $\partial C/\partial S = N(d_1)$. The delta of a put is $\partial P/\partial S = N(d_1) - 1$. Option deltas are also frequently called hedge ratios. Delta describes the change in the value of the option, given a small change in the value of the underlying security, all else equal.

Some insight into delta is gained by a graphical analysis. In Figure 19.3, the value of a call on a non-dividend-paying stock is graphed as a function of the stock price. The call's delta is the slope of the call pricing line at any point. Thus, if the call is deep out of the money, its delta is about zero. The delta of a call increases as the stock price increases. When the call is deep in the money, it sells for about its intrinsic value and the delta of the call approaches one. The delta of an American call that sells for exactly its intrinsic value is one. This should occur only when there is an ex-dividend date before expiration.

The delta of an in-the-money call will typically be above 0.50. On the expiration date, an in-the-money call will be priced on the S–K line, which is a 45° straight line. That is, the delta of an in-the-money call at expiration is 1. Thus, we can state that as the expiration date nears, the delta of an in-the-money call rises toward 1, all else equal. Similarly, the delta of an out-of-the money call will usually be below 0.50, and it will fall toward zero as time passes, all else equal. At expiration, the delta of an out-of-the-money call is 0.

Figure 19.4 shows how American puts are valued as a function of the price of the underlying asset. The delta of a put is the slope of the put pricing line at any stock price. When $S \gg K$, a put is deep out-of-the-money, and its delta is about zero. As the stock price falls, delta declines. A deep in-the-money put will have a delta of -1. The American put price curve eventually coincides with the K–S line. As shown in Figure 19.4, S^* is the critical stock price. At and below S^*, the American put should be exercised early, and its delta will have declined to -1.0. The delta of a European put will not become -1.0 until its price has declined to equal $K(1+r)^{-T}-S$.

The delta of an out-of-the-money put will rise toward zero as time passes, all else equal. The delta of an in-the-money put will fall toward -1 as the expiration date nears, all else equal. A useful rule of thumb is that at-the-money call deltas are about 0.50, and at-the-money put deltas are about -0.50.

Understanding delta is fundamental to understanding what you are buying or selling when you trade options. The purchase of a call with a delta of 0.50 is tantamount to buying half a share of stock, mostly with borrowed funds. Buying a put with a delta of -0.20 is essentially equivalent to selling short 0.20 share of stock and lending the proceeds of the short sale plus some additional sum.

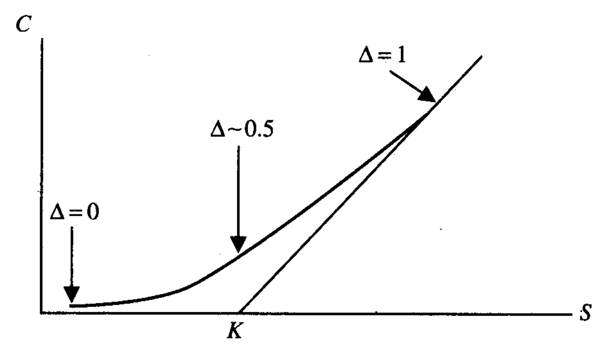

Figure 19.3 A call's delta ranges from about 0 (for a deep out-of-the-money call) to about 1.0 (for a deep out-of-the-money call) as the stock price increases.

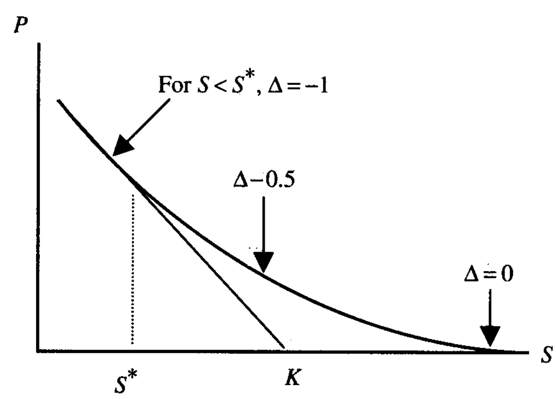

Figure 19.4 The delta of an American put declines from about 0 (for a deep out-of-the-money put) to -1.0 (for a deep out-of-the-money puts for which $S<S^*$ as S declines, where S^* is the critical stock price at which an American put sells for its intrinsic value).

Delta measures the investor's exposure to changes in the price of the underlying asset. If a call has a delta of 0.70 and if the price of the underlying asset increases by one dollar, the price of the call will likely increase by 70 cents ($70 for a call on 100 shares), all else equal. As the price of the underlying asset, its volatility, the time to expiration, or the riskless interest rate changes, so does the delta of an option.

19.2.2 What Is Gamma?

An option's gamma, Γ, measures how delta changes as the stock price changes. Recall that the delta of a put equals the delta of a call minus one. Thus, because the differentiable terms in a call delta and a put delta are the same, the gamma for a put equals the gamma for a call:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

Consider a deep out-of-the-money call. The delta of such a call is about 0. If the stock price were to change by a small amount, delta would still be 0. Thus, the gamma for a deep out-of-the-money call is about 0. Similarly, because the delta of a deep in-the-money call is about 1.0, delta will not change if S changes by a small amount. Thus, the gamma of a deep in-the-money call is also about 0. Similar logic applies for the put gamma.

The gamma for a call and a put is maximized when the option is at the money. In other words, if S changes by a small amount, the delta of an at the money call will change by a great amount. How great? If the call has a long time to expiration, the change in delta will not be much, perhaps gamma will equal 0.1 (this value will still be greater for at the money calls than for in-the-money or out-of-the-money calls with the same time to expiration). If the call has a short time to expiration and is at the money, gamma will be higher. Imagine an at-the-money call late on its expiration day. If the stock rises by one cent, the call is in the money and its delta will be 1. But if the stock declines by a penny, the call is out of the money and its delta will be 0. What this means is that deltas of at-the-money calls with short lives are quite unstable. Thus, the gammas of calls (and puts) that are exactly at the money on their expiration day are very high.

Figures 19.5 and 19.6 depict how a call's delta and gamma are functions of the price of the underlying asset, all else constant.

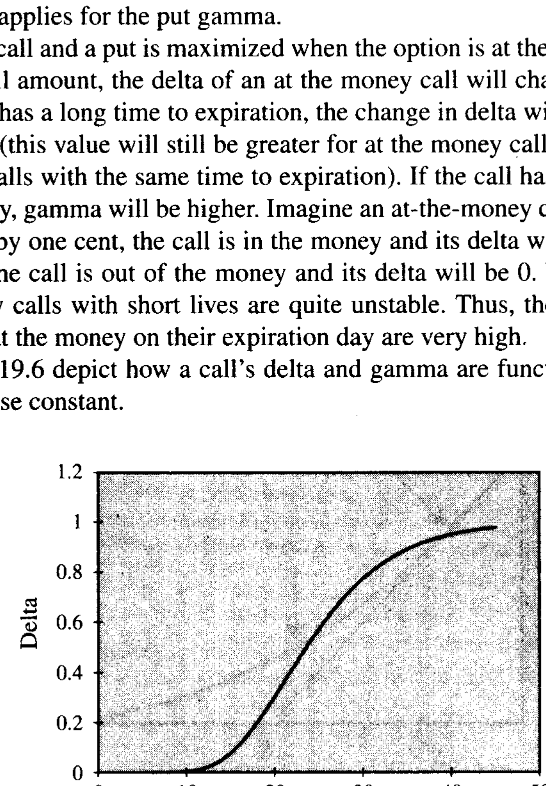

Figure 19.5 Call delta as a function of stock price.

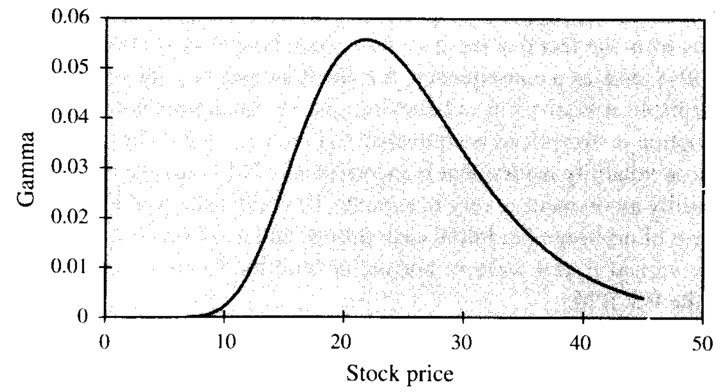

Figure 19.6 Call gamma as a function of stock price.

19.3 RISKLESS HEDGING

One of the great insights of Black and Scholes (1973) was the realization that by holding the proper combination of stock and calls, an investor could create a riskless hedge. This statement is nothing more than a restatement of the idea that a call (or any option or portfolio of options) can be replicated with a portfolio of stocks and riskless bonds. In other words, we know that

$$C = \Delta S + B$$

where $0 \leq \Delta \leq 1$ and $B \leq 0$. The equation says that buying a call is equivalent to buying stock and borrowing. Now, rearrange the equation to read

$$C - \Delta S = B$$

or

$$-C + \Delta S = -B$$

This says that buying Δ shares of stock and selling a call is like riskless lending.

The delta of a call defines a "hedge ratio" for creating a riskless hedge. If Δ shares of stock are bought and one call written, the position is riskless. Strictly speaking, the riskless hedge exists only for small changes in the stock price and over very small time intervals. As time passes and/or the stock price changes, the delta of the call changes (as measured by gamma). As Δ varies, shares of stock must be bought or sold to maintain the riskless hedge.

Now, suppose you were to find what you believe is a mispriced option. You can use the riskless hedging concept to "arbitrage." Quotes are used because the arbitrage profit will be realized only if you are correct in the estimate of volatility that went into your theoretical option price. Recall that the BSOPM assumes that volatility and interest rates are constant for the life of the option. Thus, unexpected changes in volatility and/or interest rates create risk. Note that these changes can work in your favor or against you.

Suppose you found a call option that you believed was overpriced in the market. To take advantage of your opinion, you would sell the call and buy Δ $(0 \leq \Delta \leq 1)$ share of stock. Similarly, if, in your opinion, a call was underpriced, you would buy the call and hedge the purchase by selling a fractional amount (Δ) of a share of stock. In either case, your "riskless hedge" will provide you with a rate of return in excess of the riskless interest rate, regardless of what the stock does.[7]

The term "riskless hedge" appears in quotes because, in reality, there is still risk with this strategy. The risk stems from the fact that the fractional share bought or sold is given by Δ. In turn, Δ is based on the BSOPM, and, as a consequence, Δ reflects an estimate for volatility, σ. Believing that an option is underpriced is equivalent to believing that the implied volatility (IV) is too low while believing that an option is overpriced is equivalent to believing that its implied volatility (IV) is too high. A trader whose volatility assessment is incorrect will fail to realize the profit he expected. Of course, if his volatility assessment is very inaccurate, he could even lose money.

A "purer" form of arbitrage (an initial cash inflow, and a zero cash flow at expiration, regardless of S_T) can be earned if you were to borrow or lend the funds needed to set up the riskless hedge. Compare the BSOPM:

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

to the BOPM:

$$C = S\Delta + B$$

In each equation, the first term on the right-hand side defines the amount of money to invest in the stock: $N(d_1)$ shares $=\Delta$ shares of stock, at a price of \$$S$/share. The second term on the right-hand side defines the appropriate amount to borrow or lend, as necessary, to complete the replication of the call: $B=-Ke^{-rT}N(d_2)$. You would borrow when, as in Example 19.2, you are buying Δ shares of stock and selling an overpriced call. You lend when selling Δ shares of stock and buying an undervalued call. It is important to note that what you are actually doing is either buying an undervalued call and, at the same time, selling a replicated call; or selling an overvalued call and, at the same time, buying a replicated call.[8]

EXAMPLE 19.2[9] Consider an option trader who believes that the return volatility of Applied Materials (AMAT) will decrease over the 70 days remaining before an option expiration date. Suppose AMAT is currently selling for 86 3/8, and the appropriate risk-free borrowing and lending rate is 6%. There exists a call option with a strike price of 85 with an observed price of \$7.25. At this price, the market's expectation of AMAT's return volatility until expiration (i.e., the implied volatility) is about 40.5% (use the FinCAD function aaBS_iv to verify this). However, the trader believes that the market's estimate of volatility is too high and that a volatility of about 32.7% is more accurate. Thus, in the opinion of the trader, the AMAT 85 call option should theoretically sell for about \$6.127. The trader can conduct an "arbitrage" to try to profit from his expectations. The arbitrage involves borrowing to buy the appropriate number of shares of AMAT and selling the appropriate number of "overpriced" call options. Put another way, the cheap replicated option is bought with $\sigma=32.7\%$ and dynamically adjusted as time passes and prices change, and the equivalent overpriced call is sold with $\sigma=40.5\%$. If the investor is correct in his volatility assessment, he will profit.

Column one in Table 19.1 is the days until call expiration. The second column is AMAT share prices. Following that is the weekly return and the delta for a call option with an 85 strike computed using the trader's belief that $\sigma=32.7\%$. For ease of presentation, Table 19.1 provides weekly information.[10]

Note that in Table 19.1 there are two entries at expiration. In Case I, the AMAT final stock price is such that the trader's volatility estimate 10 weeks earlier turns out to

TABLE 19.1 Replicating a Call: Delta Changes as the Stock Price Changes and as the Expiration Date Nears

Days to Expiration	AMAT Price	Weekly Return[1]	Delta of an 85 AMAT Call ($\sigma = 32.7\%$)
70	86.375		0.604
63	89.500	0.0355	0.700
56	94.000	0.0491	0.822
49	93.125	−0.0094	0.813
42	90.250	−0.0314	0.745
35	85.375	−0.0555	0.560
28	86.625	0.0145	0.620
21	93.125	0.0724	0.894
14	93.750	0.0067	0.945
7	90.625	−0.0339	0.928
Case I: Closing Price Such That Trader's Volatility Estimate was Correct[2]			
0	97.625	0.0744	1.000
	Average weekly return	0.0122	
	Weekly return standard deviation	0.0453	
	Annualized	0.3270	
Case II: Closing Price Such That Market's Volatility Estimate was correct[3]			
0	80.375	−0.1200	0.000
	Average weekly return	−0.0072	
	Weekly return standard deviation	0.0561	
	Annualized	0.4047	

[1] This is the continuously compounded rate of return, $\ln(P2/P1)$. Thus, $\ln(89.5/86.375) = 0.0355$, and $\ln(90.625/93.75) = -0.0339$.

[2] Note that if the expiration day closing price is 85, the trader's estimate of volatility (32.7%) will also be correct.

[3] Note that if the expiration day closing price is 103.5, the market's implied volatility (40.5%) will also be correct.

be correct (i.e., 32.7%). In Case II, the AMAT final stock price is such that the market's implied volatility estimate 10 weeks earlier turns out to be correct (i.e., 40.5%).

Let us return to the moment in time with 10 weeks to option expiration. Suppose a trader wants to try to arbitrage the overvalued (in his opinion) AMAT 85 call. Although the market price of this call is \$7.25, the trader believes that the call option should sell for \$6.127 because he thinks the market has overestimated AMAT's future return volatility. To assist in the understanding of the arbitrage, recall the BSOPM,

$$C = SN(d_1) - Ke^{-rT}N(d_2)$$

long call = long stock − short T-bills (i.e., borrowing)

That is, an actual long call position equals a theoretical portfolio consisting of a long stock position funded by a short position in T-bills (i.e., borrowing). In this arbitrage case, because the trader writes overpriced calls, he must "hedge" this position with a purchase of a theoretical (replicated) call. The trader replicates the call using his belief that σ is only 32.7%. Thus, the cost of replicating the call (6.127) is less than the price he receives from selling the overpriced call (7.25). The difference ($1.123) is the theoretical arbitrage profit, if the trader is correct about his volatility beliefs. Thus, with 70 days to expiration, the trader sells a call for $7.25, and borrows money ($46.043) to finance the purchase of $\Delta = 0.604$ shares of stock. At the time of the arbitrage, the trader realizes a cash inflow of $1.123:

Sell (overpriced) call	+$7.25	
Buy 0.604 shares of stock at $86.375/share:	–$52.17	Buy replicated
Borrow	+$46.043	call
Net inflow	+$1.123	

If the trader is correct, the "mispricing" she has identified has an expected profit of $1.123 per share (before commissions).

In this example we assume that readjustment of the equivalent (or "hedge") portfolio occurs every seventh day. In practice, there is a trade-off between letting the required composition of the equivalent portfolio stray too far from the current position and incurring the transactions costs to rebalance. Here, every seventh day, the trader takes one of two actions, either purchasing more shares and financing the purchase by increasing the amount borrowed, or selling shares and investing the proceeds (i.e., reducing the amount borrowed). The details of rebalancing the hedge portfolio appear in Table 19.2. The first column is the days until expiration, and the second column is the AMAT stock price. Column three is the delta of an AMAT 85 call (calculated by using the trader's volatility estimate of 32.7%). The fourth column shows how many shares are bought or sold, column five represents the cost of additional shares or the proceeds from selling shares. Column six is the interest expense or income from the subsequent share transactions.

Suppose the trader decides to sell 100 AMAT 85 call options with 70 days to expiration, each on 100 shares of stock. The trader initially hedges a short position in the calls by purchasing 6040 shares. The number of shares to purchase is obtained by multiplying the initial hedge ratio (i.e., Δ) by the number of shares underlying the options i.e., (10,000). To complete the synthetic long call position, the trader must also borrow $460,430. The interest expense will be $\$460{,}430 \times (e^{(0.06)(70/365)} - 1) = \$5329$.

Seven days later, $\Delta = 0.700$ for the 85 call. Given this new delta, the equivalent portfolio requires that the trader be long 7000 shares. Because the trader originally bought 6040 shares on day 0, this indicates a need to purchase 960 additional shares at the new AMAT market price of $89.50/share. This will cost ($\$89.50 \times 960 =$) $85,920. The trader finances the purchase by borrowing this amount at 6% for 63 days. Thus, the trader will (ultimately) incur an interest expense of $894 [$(e^{(0.06)(63/365)} - 1)\,\$85{,}920 = \$894$] on this new borrowing. As Table 19.2 illustrates, this rebalancing process occurs every seven days until option expiration.

In Table 19.2, there are two cases at expiration. Case I has the AMAT stock price at expiration that is consistent with the trader's forecast for return volatility, and the call finishes in the money. Case II has the price at expiration that is consistent with the implied volatility 70 days earlier, and the call finishes out of the money. Note that in Case I, the delta of the option increases to 1.000 (from 0.928) while the delta plummets to zero in Case II because the call has finished out of the money. In both cases, all the shares that had been purchased as part of the option replication strategy (3240) are sold on the expiration day of the call option. Now, let us use Table 19.3 to show the profitability of the trader's strategy under Case I and Case II.[11]

In Case I, the realized profit of $10,961 is very close to the expected profit of $11,233. But why was the realized total *loss* of $22,869 different from the total profit projected in Case II? The problem lies in the decision to rebalance every seventh day. As the option's expiration day nears, Δ becomes more volatile (Γ is high), particularly for around-the-money options such as this one. It is necessary to rebalance more frequently when Γ is high. The lesson to be learned is that the riskless hedge requires monitoring and frequent adjustment. The more volatile Δ is, the more frequently you should adjust. In general, traders who delta hedge should close out their positions early when gamma is high. Also bear in mind that changing interest rates will affect option prices and deltas, and that the underlying asset can be more or less volatile than originally believed. An option's delta will change not only as S and T change, but also if there are changes in σ and/or the riskless interest rate. Finally, the example ignored commissions. All these factors introduce risk and costs into the "riskless" hedge.

TABLE 19.2 Delta Hedging for Initial Position

Days to Exp.	AMAT Price	Δ for $K=85$ Call ($\sigma=0.327$)	Shares Bought (Sold)	Share Proceeds (Cost)	Interest Income (Expense)
63	89.500	0.700	960	(85,920)	(894)
56	94.000	0.822	1220	(114,680)	(1061)
49	93.125	0.813	(90)	8,381	68
42	90.250	0.745	(680)	61,370	425
35	85.375	0.560	(1850)	157,944	911
28	86.625	0.620	600	(51,975)	(240)
21	93.125	0.894	2740	(255,163)	(882)
14	93.750	0.945	510	(47,813)	(110)
7	90.625	0.928	(170)	15,406	18
Case I					
0	97.625	1.000	(3240)	316,305	0
Case II					
0	80.375	0.000	(3240)	260,415	0

TABLE 19.3 Profitability of Trader's Strategy for Cases I and II

	Intrinsic Values	
Initial Price-Expiration	**Case I**	**Case II**
Profit (Loss) on call position (Initial price – exp. day intrinsic value)[1]	(53,750)	72,500
Profit (Loss) on initial stock position (ending price – purchase price)[2]	67,950	(36,240)
Interim gain (loss) from trading shares (Sum of column 5, Table 19.3)	3,855	(52,035)
Total interest income (expense) (Sum of column 6, Table 19.3)	(1,765)	(1,765)
Financing original position (interest on $460,430 for 70 days at 6%)	(5,329)	(5,329)
Realized total profit (loss)	$10,961	($22,869)
Projected total profit [($7.25 – 6.127) × 10,000]	$11,233	$11,233

[1]The initial proceeds in both cases is $72,500 because 100 calls are sold at a price of $725 per call. The expiration day value of each call in Case I is $1262.50 because $C_T = S_T - K = 97.625 - 85 = 12.625$, and $72{,}500 - 126{,}250 = -53{,}750$. The calls expire worthless in Case II.

[2]6040 shares were initially bought at $86.375/share. In Case I, these shares are sold for $97.625/share. In Case II, they are sold at a price of $80.375/share.

Note that it is not necessary to hold onto the arbitrage portfolio for the entire life of the option. Indeed, when an option has a short time to expiration and is near the money, so that gamma is high, the arbitrage trades should be unwound. Failure to unwind early created the problem for our trader in Case II. Moreover, the trader should unwind her trades, and maybe even reverse them in the event that the option over- or under valuation is ever reversed. Here, initially, the option was overvalued at $7.25. Suppose it remained overvalued until 21 days before expiration, at which time it became undervalued. From the BSOPM, the theoretical value of this call with 21 days to expiration, σ of 0.327, a riskless rate of 6%, and a stock price of $93.125, is $8.80 (use the FinCAD function aaBS to verify this). Suppose the actual call price with 21 days to expiration was $8.00 (i.e., it became undervalued). Then, with 21 days to expiration, the trader could close the position and realize a profit of almost $39,000:

Profit (loss) on call position	(7,500)
Profit (loss) on initial stock position	40,770
Interim gain (loss) from trading shares	9,980
Total interest income (expense)	(739)
Financing original position (Interest on $460,430 for 49 days at 6%)	(3,724)
Realized total profit	$38,787

19.4 Position Deltas and Gammas

19.4.1 Position Deltas

The position delta determines how much a portfolio changes in value if the price of the underlying stock changes by a small amount. The portfolio might consist of several puts and calls on the same stock, with different strikes and expiration dates, and also long and short positions in the stock itself. The delta of one share of stock that is owned equals +1.0. The delta of a share of stock that is sold short is −1.0.

Assuming that each option covers one share of stock, the position delta Δ_Π is calculated as a weighted sum of individual deltas. That is,

$$\Delta_\Pi = \sum_{i=1}^{N} n_i \Delta_i$$

where n_i is the number of options of one particular type, or the number of shares of stock. The sign of n_i is positive if the options or stock is owned, and negative if the options have been written or the stock sold short. The delta of the ith option or stock is given by Δ_i.

Position deltas measure the change in the value of a portfolio, given a small change in the value of the underlying stock. Knowing your portfolio's position delta is as essential to intelligent option trading as knowing the profit diagrams of your portfolio. For example, suppose you buy a vertical spread using around-the-money calls. From Chapter 15, you know that you will profit if both calls finish in the money. Both calls will be in the money if the stock price at expiration is above the higher of the two strikes. Thus, the time T cash inflow equals the difference in the strike prices, $K_H - K_L$. However, the position delta for the vertical spread might only be 0.30 when the spread is purchased. The position delta for a 1:1 vertical spread equals the difference in the two

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EXAMPLE 19.3 Suppose you have positions in the following assets:

Long or Short, Number of Options or Shares	n_i	Asset	Delta/Unit	Total Deltas $= n_i \Delta_i$
Long 300 shares	+300	Stock	1.00	300.00
Long 40 contracts	+40	Puts	−0.46	−18.40
Short 150 contracts	−150	Calls	0.80	−120.00
Long 62 contracts	+62	Calls	0.28	17.36
				$\Delta_\Pi =$ 178.96

In this example, the position delta of 178.96 is positive. This means that if the stock price were to increase by one dollar, the value of this portfolio would rise by \$178.96. If the stock price were to decline by one dollar, the value of this portfolio would fall by \$178.96. This example assumes that the underlying asset of an option is one share of stock.

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calls' deltas. Thus, if the stock price were to rise or fall by one dollar immediately after purchasing the spread, your profit or loss is only 30 cents.

A *delta-neutral* position is a portfolio that is immune to changes in the stock price. That is, the portfolio of options and stock has a position delta of 0.0. However, arbitrageurs are not the only users of delta-neutral positions. Many market makers trade options and try to maintain a delta-neutral position. These market makers are not speculating on movements of the stock price. Instead, they can concentrate on capitalizing on the bid–ask spread and trade what they perceive to be mispriced options. Being delta neutral allows them to sleep at night. Delta-neutral hedging is not totally effective, Since there is always a chance that the stock will jump unexpectedly. Recall that delta estimates the change in the value of an option for a *small* change in the price of the underlying asset. A large change in the price of the underlying asset will create a change in the value of a delta-neutral portfolio.

Many institutions are also delta hedgers, including investment bankers and commercial bankers, who sell nontraded (over-the-counter) options to their clients or purchase them from their clients. For example, a customer who owns a large block of shares of a stock on which no options trade might wish to write covered calls or buy protective puts on those shares. As another example, consider a corporate client who anticipates the purchase (or sale) of a number of government bonds, or some foreign exchange, in the near future. This client would like to insure that the firm pays a price for the purchase of the asset that is no higher than some maximum tolerable amount, or to insure that the price it will receive for the bonds or foreign exchange is no lower than some tolerable minimum.

These are examples of customers who would like to modify the risk exposure they face in some way. The bank can buy or sell custom-designed options to the client at a price that reflects the current price of the underlying asset, the riskless interest rate, the strike price, the time to expiration, and the estimated volatility of the underlying asset *plus a profit premium*. Then, the bank will enter into a delta-neutral position by trading the underlying asset and riskless securities, reflecting the underlying asset's estimated volatility. As long as the estimated volatility is correct, the bank will earn a profit by providing the service to the customer. The bank will often also be willing to allow the customer to offset his position early, for a price, of course.

19.4.2 Position Gammas

Similar to position deltas, a position gamma Γ_Π is the weighted sum of the gammas of the elements of the portfolio:

$$\Gamma_\Pi = \sum_{i=1}^{N} n_i \Gamma_i$$

where n_i is the number of options of one particular type. The sign of n_i is positive if the option is owned, and negative if the option has been written. The gamma of the ith option is given by Γ_i. Note that the gamma of a stock is zero, because the delta of a stock is always to one.

Ideally, a delta-neutral hedger would like to maintain a portfolio with a positive gamma. Recall from Sections 19.1.1 and 19.1.2 that the gamma of both a call and a put is given by the same formula and that gammas cannot be negative. However, a *portfolio* can have a positive or negative gamma. It is important to note that a portfolio with a positive gamma increases in value if the underlying stock value *changes*. Further, a portfolio with a negative gamma decreases in value if the underlying stock value *changes*.

EXAMPLE 19.4 Suppose $S=K=\$100$, $r=8\%$, $\sigma=30\%$, and $T=180$ days. Then, by using the BSOPM, one can generate the following information.

Variable	Call Option	Put Option	Stock
Price	\$10.30	\$ 6.44	\$100
Delta	0.6151	−0.3849	1
Gamma	0.0181	0.0181	0

Now, suppose a trader is considering the use of either calls or puts to maintain a delta-neutral portfolio. This can be accomplished with the purchase of puts or by writing calls. The impact of a positive or negative gamma can be seen by the following information, presented on a per-share basis.

Using Puts

Delta-neutral strategy: long 1 put at \$6.44; long 0.3849 share at \$100/share.

Positive position gamma: $(1 \times 0.0181) + (0.3849 \times 0) = 0.0181$.

Stock Price	Put Price	Value of 0.3849 Share	Portfolio Value	Percent Change (from $S = 100$)
90	11.25	34.64	45.89	2.14
100	6.44	38.49	44.93	
110	3.41	42.34	45.75	1.83

Using Calls

Delta-neutral strategy: short 1 call at \$10.30; long 0.6151 share at \$100/share.

Negative position gamma: $(-1 \times 0.0181) + (0.3849 \times 0) = -0.0181$.

Stock Price	Call Price	Value of 0.6151 Share	Portfolio Value	Percent Change (from $S = 100$)
90	5.12	55.36	50.24	−1.89
100	10.30	61.51	51.21	
110	17.28	67.66	50.38	−1.62

Note that when the delta hedger uses puts to construct a portfolio, the value of the portfolio increases, regardless of the direction of the change in the underlying asset price. When calls are used, however the portfolio value falls whether the underlying asset price increases or decreases.

From a cost standpoint, a delta-neutral hedger would like to have a portfolio with a low, but positive gamma. Recall that gamma measures changes in delta, and that deltas change as S, T, σ, and/or r change. Thus, a portfolio that is delta-neutral today may not be delta neutral tomorrow. A low position gamma will mean that the delta-neutral investor will conserve on the transactions costs of readjusting his portfolio delta back to zero if S changes. But a positive gamma at least compensates a trader for bearing the risk of fluctuating delta, as the value of his portfolio will increase if the underlying asset's price changes, all else equal.

Note that like most concepts dealing with options, delta and gamma hedging (and vega hedging discussed next) are calculus concepts that work best when there are very small changes in the price of the underlying asset.

Example 19.6 Delta–Gamma–Vega Hedge In Equation (19.4), an important effect has been assumed to be constant. This effect, which stems from the volatility of the underlying asset, is known as "vega."

Now suppose $S=K_1=\$100$, $r=8\%$, $\sigma=30\%$, $T=180$ days, $K_2=110$, and $K_3=90$. Then, one can use the BSOPM to generate the following information.

	Call Option			
Variable	**$K=90$**	**$K=100$**	**$K=110$**	**Stock**
Price	\$16.33	\$10.30	\$6.06	\$100
Delta	0.7860	0.6151	0.4365	1
Gamma	0.0138	0.0181	0.0187	0
Vega	0.2046	0.2684	0.2766	0

Suppose the trader is long 200 call options with the 100 strike. Each option is on 100 shares of stock. Because the trader wants to create a portfolio that is delta-gamma-vega neutral, he must simultaneously solve for the number of shares to purchase, S, the position in the 110 calls, K_{110}, and the position in the 90 calls, K_{90}. That is, he must choose the position in these securities so that the portfolio's delta, gamma, and vega all equal zero (at least initially). To solve this problem, we begin by noting that the gamma and the vega of the stock are zero and that the position in the option with the 100 strike is given (i.e., $K_{100}=-200$). Thus, to be gamma neutral and vega neutral implies

$$0=(-200\times 100\times 0.0181)+(S\times 0)+(K_{110}\times 100\times 0.0187)+(K_{90}\times 100\times 0.0138)$$

and

$$0=(-200\times 100\times 0.2684)+(S\times 0)+(K_{110}\times 100\times 0.2766)+(K_{90}\times 100\times 0.2046)$$

This becomes:

$$362=1.87K_{110}+1.38K_{90}$$
$$5368=27.66K_{110}+20.46K_{90}$$

Solving these two equations simultaneously yields $K_{110}=14.76$ and $K_{90}=-282.33$ for any value of S. Then, solving for the number of shares is accomplished by noting that the delta of a share is 1 and that the trader also wants to hold a delta-neutral portfolio. Accordingly,

$$0=(-200\times 100\times 0.6151)+(S\times 1)+(14.76\times 100\times 0.4365)+$$
$$(-282.33\times 100\times 0.7860)$$

yields $S=9244.86$. Assume the trader buys 9245 shares (and buys 15 of the $K=110$ calls and sells 282 of the $K=90$ calls). To examine the performance of the delta–gamma–vega hedge, suppose an earnings announcement makes the volatility of the underlying asset jump to 35 or 25%. Letting $r=8\%$, $T=180$ days and $S=100$, we have the following.

Stock Volatility	Call Price $K=90$	$K=100$	$K=110$	Portfolio Value	Percent Change in Portfolio Value (from $\sigma=30\%$)
25%	15.36	8.97	4.68	$677,763	−0.1938%
30%	16.33	10.30	6.06	$679,084	
35%	17.39	11.65	7.45	$678,277	−0.1188%

Therefore, the portfolio, as constructed, hedges the portfolio against instantaneous changes in the volatility of the underlying asset. It is important to note, however, that a delta–gamma–vega hedge may outperform or underperform a delta–gamma hedge when the underlying stock price changes.

19.4.4 Time Spreads

Time spreads are also sometimes called calendar spreads or horizontal spreads. Time spreads are identified by the ratio of options written and purchased.

19.4.4.1 1:1 Time Spreads

To create a 1:1 time spread, the investor buys one option and sells one option. The two options must be either both calls or both puts, and must share the same strike price on the same underlying asset. However, unlike most of the option portfolios discussed thus far, the time spread portfolio of options has options with different maturity dates.

Traders like to hold long positions in time spreads. A long time spread position involves selling the option with a short time to expiration and buying the longer term option. This is done for two reasons. First, the trader typically wishes to exploit the higher theta of the nearby option. Options, particularly if they are at the money, will usually lose any time value at a fast and increasing rate as the expiration date nears. At-the-money options usually have more time value than other options, all else equal. Thus, a hypothetical at-the-money call with one month to expiration might be sold for $2, and a call with two months to expiration might be bought for $3. If the stock price remains unchanged, the former will expire worthless (creating a profit of $2), and the latter (which now has one month to expiration) might be sold for $2 (creating a loss of 1). Thus, the net profit is $1. The second reason for the tendency of traders to sell the nearby option is sold is that many empirical studies have found that options with short lives are usually overvalued relative to options with a longer time to expiration, all else equal.[14]

The maximum profit on a time spread occurs when the stock price equals the strike price on the expiration date of the nearby call. The short-term written option expires worthless, and the longer term option can be sold with some time value remaining. An important risk to be kept in mind is that if the written, short-term option is in the money as its expiration date nears, it could be exercised early. To preclude this possibility, therefore, in-the-money time spreads using American options often must be closed out early.

EXAMPLE 19.7 Time Spread On August 1, the following are observed:

$S = 42.875$

$K = 45$

$r = 11\%/\text{year}$

Price of a call expiring on September 21 = 2

Price of a call expiring on December 21 = 3.375

Prepare the profit diagram for a long time spread.

Solution

On August 1:

Sell 1 Sep 45 call	+2
Buy 1 Dec 45 call	−3.375
	−1.375

Thus, the initial outlay for a call on 100 shares is $137.50.

The profit table is prepared based on the closing prices of September 21. Using the BSOPM to estimate the September 21 prices of the December call given a range of stock prices on September 21. A volatility of $\sigma = 40\%$ is used because this is the implied volatility of the September call on August 1. Finally, remember that on September 21, the December call will have a time to expiration of 0.25 year.

	On September 21		Time T	Profit/Loss
S_T	Buy Sept 45 call	Sell Dec 45 call	Cash Flow, CF_T	CF_0+CF_T
37	0	+0.893	+0.893	−0.482
38	0	+1.143	+1.143	−0.232
39	0	+1.436	+1.436	+0.061
40	0	+1.775	+1.775	+0.400
41	0	+2.161	+2.161	+0.786
42	0	+2.594	+2.594	+1.219
43	0	+3.074	+3.074	+1.699
44	0	+3.601	+3.601	+2.226
45	0	+4.173	+4.173	+2.800
46	−1	+4.788	+3.794	+2.419
47	−2	+5.450	+3.450	+2.075

48	−3	+6.144	+3.144	+1.769
49	−4	+6.874	+2.874	+1.499
50	−5	+7.637	+2.637	+1.262
51	−6	+8.429	+2.429	+1.054
52	−7	+9.248	+2.248	+0.873
53	−8	+10.091	+2.091	+0.716
54	−9	+10.956	+1.956	+0.581
55	−10	+11.840	+1.840	+0.465

The profit diagram for the long time spread appears in Figure 19.7.

The maximum loss is of \$1.375 is realized if the stock price plummets until both options are worthless on September 21, or if the stock price rises so high that both options sell for their intrinsic value ($S_T - K$) on September 21. However, the range of stock prices for which a profit is earned is quite wide: from \$39/share to a price above \$55/share.

The strategy described in Example 19.7 is usually called a bullish time spread because the stock price must rise for the maximum profit to be reached. If both calls were in the money, a bearish time spread would be purchased.

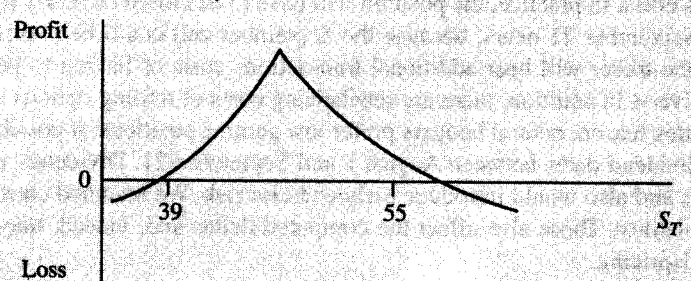

Figure 19.7 Profit diagram for a long time spread.

19.4.4.2 Neutral Time Spreads

A neutral time spread is a strategy designed to capitalize on two options that are somehow mispriced relative to each other. The two options differ only in their times to expiration. As discussed earlier, usually the short-term option is written and the longer option is purchased. The number of each option traded is designed to create a delta-neutral portfolio.

In Example 19.7, the implied standard deviations was 0.4043 for the September call and 0.3262 for the December call. Thus, assuming that the BSOPM is the proper pricing formula, the short-term call is overvalued relative to the December call.

Assume that the "true" volatility is 0.40/year. Then the hedge ratios of the two options are $\Delta = 0.4439$ for the September call and $\Delta = 0.5415$ for the December call. A delta-neutral portfolio of these two options is

$$\Delta_p = 0 = n_1\Delta_1 + n_2\Delta_2$$

Rearranging, we have

$$\frac{n_1}{n_2} = \frac{-\Delta_2}{\Delta_1}$$

Using these deltas, and naming the September and December calls as assets 1 and 2, respectively, we write

$$\frac{n_1}{n_2} = \frac{-0.5415}{0.4439} = -1.22 \tag{19.6}$$

Equation (19.6) says that to create a delta-neutral position, 1.22 September calls should be sold for each December call purchased. This allows the investor to arbitrage the relative mispricing of the two options and not speculate on price movements. As time passes and as the price of the stock changes, the deltas will change, and the relative number of options needed to maintain the delta neutral position also will change. Thus, the position must be monitored. The investor typically rebalances the portfolio when it is necessary to buy or sell some threshold number of options to maintain a delta-neutral position. When rebalancing, the investor should, of course, always buy the more undervalued option, or sell the more overvalued option.

As usual, there are real risks and costs that must be included. For example, we have ignored transactions costs. In practice, the position will have to be closed out early if the options are in the money as September 21 nears, because the September call could be exercised early. If they are exercised, the trader will bear additional transactions costs of having to purchase the stock and making delivery. In addition, there are rebalancing costs of trading options to maintain delta neutrality. For this reason, neutral hedgers prefer low gamma positions, if possible. We assumed there are no ex-dividend dates between August 1 and September 21. Dividends would affect the computed deltas and also would introduce early exercise risk. We assumed constant interest rates and constant volatility. These also affect the computed deltas and, indeed, may be the source of the apparent mispricing.

19.5 Caps, Floors, and Collars: Using Options to Manage Interest Rate Risk

19.5.1 Caps

Many firms, financial institutions, and individuals borrow and/or lend money at variable interest rates. This means that the interest rate on the loan balance changes at prespecified dates. The magnitude of the change in the interest rate on the loan is typically tied to a publicly available index such as LIBOR or a commercial paper index.

A **cap** is a clause or product that effectively places a maximum interest rate on a variable-rate loan. With a cap, a borrower will pay the lesser of the rate that is pegged to the index, or the cap rate. Sometimes caps are called **ceilings**. An interest rate **floor** effectively sets a *minimum* interest rate on a variable-rate loan. A **collar** effectively sets both a maximum and a minimum interest rate on a variable-rate loan. Many variable-rate mortgages have collars. For example, a home buyer might be offered a $100,000 mortgage with an interest rate of 8.5% for the first year. Each subsequent year, a new interest rate is applied to the remaining loan balance. The new interest rate

EXAMPLE 19.8 Consider a notional principal of \$8 million with a cap of 10%. The interest rate is reset quarterly, and interest payments are made quarterly. Suppose that on one of the rate-fixing dates, the underlying index (LIBOR) is 12%.

With an annual rate of 10%, the quarterly rate is 2.50%, and the quarterly payment would be (\$8,000,000)(0.0250) = \$200,000. Instead, because the new interest rate is 12%, the actual payment based on the quarterly market interest rate is (\$8,000,000)(0.0300) = \$240,000. If the borrower had purchased a 10% cap on a notional principal of \$8 million, he would be paid the difference in these amounts, \$40,000, by the financial institution that sold him the cap. This effectively sets the maximum annual interest rate of 10%.

What this means is that if this call finishes 200 basis points in the money (because 12% is 200 basis points above the strike price of 10%), the payoff to the call owner (the borrower) is \$40,000.

might equal the prevailing rate on one-year T-bills plus 225 basis points. However the interest rate can never exceed 12.5% (a cap), or be below 5.5% (a floor).[15]

Hundreds of billions of dollars of loans in the United States have caps, floors, or collars. It is important to note, however, that caps, floors, and collars apply only to the loan payments. In a cap, floor, or collar, there is no default risk concerning the loan principal. However there is default risk concerning the loan payments. The loan principal is frequently called **notional principal**. The dollar payment with a cap and floor is a function of the difference between the market interest rate and the cap/floor rate times the notional principal.

A cap can be viewed as a call option on an interest rate. Assume that a loan has an interest rate cap of 10% (the strike price). If, on the interest rate reset date, the new interest rate is 10% or less. the call expires worthless. However, if the new interest rate is greater than 10%, the call pays off the difference between the new interest rate that would exist in the absence of the cap and 10%, times the notional principal. This amount must then be adjusted for the timing of the loan repayments at the new loan rate.

More generally, define:

NP = notional principal
R_c = interest rate cap
R = new interest rate (determined by some interest rate index, such as LIBOR)
d = days until the next interest rate reset date

Then, a cap's payoffs[16] on each interest rate reset date are as follows:

$$\begin{array}{ll} 0 & \text{if } R < R_c \\ (NP)(R - R_c)\dfrac{d}{360} & \text{if } R > R_c \end{array}$$

In Example 19.8, the cap's payoff is (\$8,000,000)(0.12 − 0.10)(90/360) = \$40,000, because 12% > 10%.

Financial institutions sell caps to their clients. The latter are usually firms that have borrowed money at variable interest rates and wish to insure against the possibility that at some future interest rate reset date, they will have to pay more than the interest rate cap.

Because most cap agreements have a series of interest rate reset dates, a cap is actually a series, or portfolio, of call options, each expiring on a reset date. The term **caplet** refers to any one of the options making up a cap.

19.5.2 Floors

An interest rate floor is a put option on an interest rate. Floors represent insurance for the lending institution that it will never receive less than the floor interest rate on a variable loan rate that it has made. If the new interest rate is above the strike (the floor), the put expires worthless. If the market interest rate (e.g.,three month LIBOR) is below the interest rate floor (the strike price), the put pays off to the lender. Borrowers sell floor agreements to lenders, and borrowers must make payments to the lending institution if interest rates decline below the floor rate.

Define the following:

NP = notional principal
R_f = interest rate floor
R = new interest rate (determined by some interest rate index, such as LIBOR)
d = days until the next interest rate reset date

Then, a floor's payoffs on each interest rate reset date are as follows

$$\begin{array}{ll} 0 & \text{if } R \geq R_f \\ NP(R_f - R)\dfrac{d}{360}; & \text{if } R < R_f \end{array}$$

19.5.3 Collars

An interest rate collar combines a cap and a floor. Equivalently, the lending institution sells an interest rate call to the borrower and buys an interest rate floor from the borrower. A collar's payoffs on each interest rate reset date are as follows:

$$\begin{array}{lll} NP\,(R - R_f)\dfrac{d}{360} & \text{if } R < R_f & \text{(borrower pays lender)} \\ 0 & \text{if } R_f < R < R_c & \\ NP(R - R_c)\dfrac{d}{360} & \text{if } R > R_c & \text{(lender pays borrower)} \end{array}$$

The profit diagram for a collar from the lender's viewpoint (Figure 19.8a) shows that if $R < R_f$, the lender profits because he owns an interest rate put. If $R > R_c$, the lender loses because he has sold an interest rate call. In Figure 19.8b, the profit diagram for a collar from the borrower's viewpoint, the borrower benefits from owning a cap (an interest rate call) if $R > R_c$. The borrower loses if interest rates decline below the floor rate.

Figure 19.9 shows that a collar effectively sets a minimum and a maximum interest rate for a variable-rate borrower who has also purchased a collar.

Typically, caps, floors, and collars are out of the money when they are originated. By choosing the cap and floor rates appropriately, the lender can create what is known as a "zero-cost collar." That is, the value of the call sold by the lender equals the value of the put the lender purchases.[17]

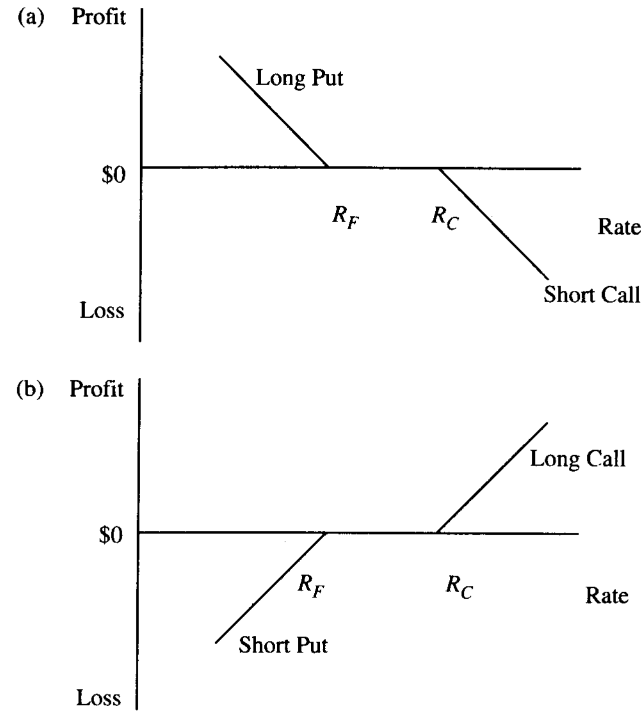

Figure 19.8 (a) A collar's payoff profile to the lender, where "Rate" is the interest rate on the reset date and R_f and R_c are the interest rate floor and cap, respectively. (b) A collar's payoff profile to the borrower.

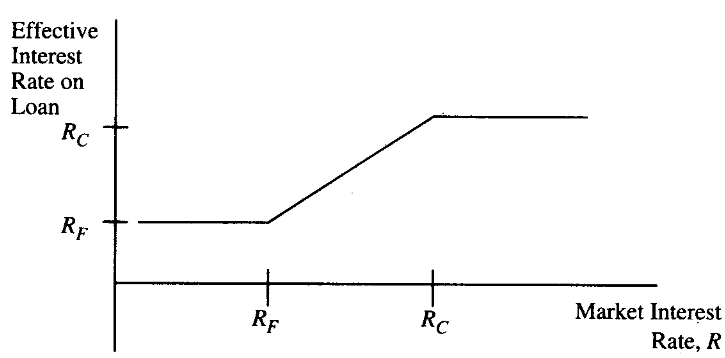

Figure 19.9 A borrower who has borrowed at a variable interest rate and has also bought a collar will never pay an interest rate below the floor rate or above the cap rate.

19.5.4 Valuing Caps, Floors, and Collars

There is a secondary, OTC, market for caps, floors, and collars. Generally, these products trade in multiples of $5 million notional principal, with maturities ranging from three months to 10 or more years. While the traded caps, floors, and collars usually have interest rates based on three-month LIBOR, they can have monthly, quarterly, or semiannual reset dates. Because financial institutions trade these products to adjust their net risk exposure, it is important to be able to value these products.[18]

Consider the following information. Suppose a two-year variable-rate loan in the amount of \$10 million exists with reset dates on March 31, June 30, September 30, and December 31. The reset rate is LIBOR. Let today be March 1, 2001. Define $r(0, t)$ as spot LIBOR from March 1, 2001 (day 0) until time t. Define $r(t1, t2)$ as the forward LIBOR from time $t1$ until time $t2$. This forward interest rate might be inferred from Eurodollar futures prices, or from current spot Eurodollar rates. Assume that the volatility of forward Eurodollar interest rates is 20%/year.

Recall that a cap, floor, or collar is a portfolio of options that expire on different reset dates. Thus, to value a cap, floor, or collar, an interest rate option pricing model is used to value each option that expires on each interest rate reset date. Although the more complex Heath, Jarrow, and Morton (1992) model yields more precise option prices, a common practice is to use a version of Black's (1976) forward option pricing model. Thus, to value a caplet or floorlet, use

$$\text{caplet value} = C = (NP)(t_2 - t_1)e^{-rt_2}[FN(d_1) - KN(d_2)] \tag{19.7}$$

$$\text{floorlet value} = P = (NP)(t_2 - t_1)\, e^{-rt_2}[KN(-d_2) - FN(-d_1)] \tag{19.8}$$

where

t_1 = the period of time until the start of the forward period = the time to expiration of the caplet, in years = the time until the loan reset date

t_2 = end of the forward period

$t_2 - t_1$ = length of the forward period

$r = (0, t_2)$ = the spot interest rate for the period ending at time $t2$

$F = r(t_1, t_2)$; thus F, the forward interest rate, is the underlying asset

K = cap or floor interest rate

σ = standard deviation of the percentage changes in the forward interest rate

$d_1 = [\ln(F/K)/\sigma t_1^{1/2}] + 0.5\sigma t_1^{1/2}$

$d_2 = d_1 - \sigma t_1^{1/2}$

Equations (19.7) and (19.8) take the values of Black's forward option pricing model and multiply them by the notional principal times the length of the forward period. The payment on the caplet is made at time t_2.

Table 19.4 presents a valuation example for caplets and floorlets. For times to expiration of less than one year, the riskless interest rate is approximately the T-bill bond equivalent yield. For expirations of more than one year, the riskless interest rate is approximated by the yield on a zero-coupon, or stripped, T-note (or T-bond). The forward rates are approximately equal to Eurodollar futures add-on yields and are presented as percentages. The latter represent F, the underlying asset, in Black's model. The notional principal is \$10 million. Today is March 1, 2001. The strike price for the cap, R_c, is 10%. The strike price for the floor, R_f, is 5%. The volatility of each underlying asset (each caplet has its own unique forward rate; each *could* have a different volatility) is 20%. Each caplet and floorlet value is given in dollars.[19]

The value of the cap equals the sum of the eight call values (the individual caplets), or \$4,314.23. By summing the values of the eight floorlets, you obtain the floor value of \$7930.66. In a collar, the lender sells the cap and buys the floor, so the cost of the collar to the buyer is \$3,616.42.

TABLE 19.4 Valuation of Caplets and Floorlets

Loan Reset Date	Time to Expiration, in Years	$r(0, t_1)$	$r(t_1, t_2)$	Cap Value for $R_c = 10\%$	Floor Value for $R_F = 5\%$
03/31/01	0.08219	0.055	6.2321	0.0000	0.0608
06/30/01	0.33151	0.062	6.2825	0.1341	139.1975
09/30/01	0.58356	0.063	6.4735	19.0599	387.8555
12/31/01	0.83562	0.064	6.6701	160.1481	594.3566
03/31/02	1.08219	0.065	6.8567	540.2025	779.6977
06/30/02	1.33151	0.066	6.4431	456.8923	2036.8952
09/30/02	1.58356	0.066	7.1376	1970.4166	1184.0161
12/31/02	1.83562	0.067	6.5372	1167.3808	2808.5782
03/31/03	2.08219	0.067			
				Sum: $4314.2343	$7930.6576

If the borrower wanted a zero-cost collar with a cap of 10%, then the lending institution would search for the interest rate floor that had a value of $4,314.23, the same as the cap.[20]

FinancialCAD can be used to solve for the value of a cap or floor. Before we do that, however, it is useful to first use aaConverR_DFcrv to convert the series of spot interest rates into a series of discount factors. This is shown in Figure 19.10. Then, in Figure 19.11, we use these discount factors and the function aaRatefwd_crv to compute the forward rates that exist in the spot yield curve. These are the same forward rates shown in Table 19.4.

We use the FinancialCAD function aaRcapBL_fs statv to solve for the value of a cap or floor, as shown in Figure 19.12. Note that the discount factors produced by aaConvertR_DFcrv (Figure 19.10) are used as in input in Figure 19.12. Other functions, such as aaRcap_BL_dgen and aaRcap_BL can also be used to value caps and floors.

There is a small difference between the total dollar value of the cap that is computed by FinancialCAD in Figure 19.12, $4330.66, and the cap value of $4314.23 generated in Table 19.4. The difference arises because FinancialCAD does not discount by e^{rt_2} like we do in equations 19.7 and 19.8. Click on FinancialCAD's "Show Math" button in its function finder for Rate Caps and Floors to learn how they discount.

19.6 SUMMARY

In this chapter, we focus on the details of how options provide insurance and some necessary technical details concerning how option prices change as the factors that influence option prices change.

The BSOPM can be differentiated with respect to each of its five parameters, K, T, r, σ, and S. In addition, the second derivative of the BSOPM with respect to S is quite useful. The results are useful formulas that predict how much an option value will change if only *one* input parameter changes by a small amount, and all other inputs are held constant.

A particularly important sensitivity is known as delta. Delta is also called an option hedge ratio. Delta measures the amount by which the option value changes, given a small change in the

aaConvertR DFcrv		
value (settlement) date	1-Mar-2001	
rate curve	t 48 4	
		annual compounding
rate quotation basis	1	actual/365 (fixed)
accrual method of rate	1	actual/365 (fixed)

t_48_4
rate curve

maturity date	yield to maturity
31-Mar-2001	0.055
30-Jun-2001	0.062
30-Sep-2001	0.063
31-Dec-2001	0.064
31-Mar-2002	0.065
30-Jun-2002	0.066
30-Sep-2002	0.066
31-Dec-2002	0.067
31-Mar-2003	0.067

discount factor curve - aaConvertR_DFcrv

grid date	discount factor
1-Mar-2001	**1**
31-Mar-2001	**0.995609058**
30-Jun-2001	**0.980256078**
30-Sep-2001	**0.964975317**
31-Dec-2001	**0.949482868**
31-Mar-2002	**0.934119588**
30-Jun-2002	**0.918419491**
30-Sep-2002	**0.903742602**
31-Dec-2002	**0.887770948**
31-Mar-2003	**0.873687812**
#N/A	**#N/A**

Figure 19.10 The financialCAD function aaConvertR_DFcrv converts a series of spot interest rates into a series of discount factors.

value of the underlying asset and holding everything else equal. Delta and gamma, the rate at which delta changes, are important factors to consider when one is managing risk.

Portfolios consisting of the underlying asset and options also have deltas and gammas. These are commonly called position deltas and position gammas. We provide an example of how risk managers can immunize their portfolios against delta and gamma effects.

We also provide a discussion on time spreads. Time spreads are portfolios of options that do not have the same time to expiration. We present an example of how such a portfolio can be delta hedged.

Because many firms, institutions, and individuals borrow at variable interest rates, we discuss the use of options to manage interest rate risk. In particular, we show how a borrower can ensure

aaRatefwd crv		
date list	t_7	
discount factor curve	t_43_1	
interpolation method	1	linear
		actual/365 (fixed)
accrual method	1	actual/365 (fixed)
margin above or below a floating rate	0	

t_7
date list
effective and terminating date pairs

31-Mar-2001
30-Jun-2001
30-Sep-2001
31-Dec-2001
31-Mar-2002
30-Jun-2002
30-Sep-2002
31-Dec-2002
31-Mar-2003

t_43_1
discount factor curve

grid date	discount factor
1-Mar-2001	1
31-Mar-2001	0.995609058
30-Jun-2001	0.980256078
30-Sep-2001	0.964975317
31-Dec-2001	0.949482868
31-Mar-2002	0.934119588
30-Jun-2002	0.918419491
30-Sep-2002	0.903742602
31-Dec-2002	0.887770948
31-Mar-2003	0.873687812

forward rate list given date list and discount factors – aaRatefwd_crv

forward rate
0.062820965
0.062825192
0.064734825
0.066700917
0.068566618
0.064430912
0.071376297
0.065372252

Figure 19.11 The financialCAD function aaRatefwd_crv, converts a series of discount factors (based on spot interest rates), into forward rates.

that he will never pay a variable interest rate above a set level, known as a cap. In addition, we show how a lender can ensure that he will never receive less on a variable rate loan than predetermined amount, known as a floor. Finally, we present an example of how to calculate the value of a cap and a floor.

aaRcapBL_fs_statv		
value (settlement) date	1-Mar-2001	
		linear
interpolation method	1	rate cap
option type	1	actual/365 (actual)
accrual method	1	actual/365 (actual)
current fixing of rate as of last reset day		
rate cap payment table - using volatility	t_3	
discount factor curve	t_43_1	

t_3

rate cap payment table - using volatility

effective date	terminating date	notional principal amount	exercise rate	implied volatility
31-Mar-2001	30-Jun-2001	10,000,000.00	0.1	0.2
30-Jun-2001	30-Sep-2001	10,000,000.00	0.1	0.2
30-Sep-2001	31-Dec-2001	10,000,000.00	0.1	0.2
31-Dec-2001	31-Mar-2002	10,000,000.00	0.1	0.2
31-Mar-2002	30-Jun-2002	10,000,000.00	0.1	0.2
30-Jun-2002	30-Sep-2002	10,000,000.00	0.1	0.2
30-Sep-2002	31-Dec-2002	10,000,000.00	0.1	0.2
31-Dec-2002	31-Mar-2003	10,000,000.00	0.1	0.2

t_43_1

discount factor curve

grid date	discount factor
1-Mar-2001	**1**
31-Mar-2001	**0.995609058**
30-Jun-2001	**0.980256078**
30-Sep-2001	**0.964975317**
31-Dec-2001	**0.949482868**
31-Mar-2002	**0.934119588**
30-Jun-2002	**0.918419491**
30-Sep-2002	**0.903742602**
31-Dec-2002	**0.887770948**
31-Mar-2003	**0.873687812**
#N/A	**#N/A**

rate cap table - Black '76 - aaRcapBL_fs_statv

effective date	terminating date	notional principal amount	exercise rate	implied volatility	fair value	delta	gamma
31-Mar-2001	**30-Jun-2001**	**10,000,000.00**	**0.1**	**0.2**	**-3.13037E-12**	**8.14E-14**	**1.82E-14**
30-Jun-2001	**30-Sep-2001**	**10,000,000.00**	**0.1**	**0.2**	**0.13427369**	**0.008424**	**0.000489**
30-Sep-2001	**31-Dec-2001**	**10,000,000.00**	**0.1**	**0.2**	**19.09125701**	**0.670891**	**0.020825**
31-Dec-2001	**31-Mar-2002**	**10,000,000.00**	**0.1**	**0.2**	**160.49948**	**3.8818**	**0.079043**
31-Mar-2002	**30-Jun-2002**	**10,000,000.00**	**0.1**	**0.2**	**541.7054766**	**9.996654**	**0.148473**
30-Jun-2002	**30-Sep-2002**	**10,000,000.00**	**0.1**	**0.2**	**458.4045613**	**8.378268**	**0.123284**
30-Sep-2002	**31-Dec-2002**	**10,000,000.00**	**0.1**	**0.2**	**1978.204807**	**25.14651**	**0.23784**
31-Dec-2002	**31-Mar-2003**	**10,000,000.00**	**0.1**	**0.2**	**1172.616137**	**16.35053**	**0.173722**
					4330.655993		

Figure 19.12 The financialCAD function aaRcapBL_fs_statv, solves for the value of a cap or floor.

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Notes

[1]Of course, tax, accounting, and legal factors may all affect the risk manager's decision to use options, futures, forwards, or swaps.

[2]Recall that in a continuous hedging strategy, the risk manager will always hedge, regardless of her beliefs or of price forecasts. In contrast, a selective hedging strategy will allow the risk manager to hedge only when she expects prices to move adversely.

[3]In addition, the second derivative of the BSOPM with respect to S is quite useful.

[4]Note that the partial derivative with respect to the strike price has no Greek name. This is most likely because the strike price of the option is not stochastic. For details on the derivations of these formulas, see Galai and Masulis (1976) and Conine and Tamarkin (1984).

[5]By convention, theta is frequently defined as the *negative* of $\partial C/\partial T$.

[6]See the discussion of theta in Chapter 14 for some intuition concerning this result.

[7]It is assumed that you can sell short the stock costlessly and receive full use of the proceeds. Also, note that under the assumptions of the BSOPM, Δ can asymptotically approach 0.0 and 1.0 but can never exactly equal those values. The tails of a normal curve stretch out to infinity, so that $N(d_1)$ can never exactly equal 0.0 or 1.0.

[8]The principles of call replication using the binomial option pricing model appear in Chapter 17.

[9]The seeds of this example stem from material presented in Natenberg (1994, Chapter 5).

[10]Note that the choice to present the data every week is arbitrary. In practice, investors monitor Δ daily, or even intra daily, and adjust their positions when their investment in the underlying asset is sufficiently different from the Δ that is prescribed by their option pricing model.

[11]Recall that there are many different possible paths through which the stock price can wander before option expiration. Vastly different results for such an example can be obtained depending on the interim volatility of the stock price (i.e. the volatility over the first nine weeks). You are encouraged to prepare a spreadsheet and experiment. One interesting aspect is that if the volatility over the first nine weeks is lower than the trader's forecast, it is possible to obtain a result closer to the projected profit than if the volatility over the first nine weeks is higher than the trader's forecast for subsequent return volatility.

[12]The reader should note that this logic applies to a wide variety of derivative securities (such as forwards, futures, and options on futures, as well as options on the underlying asset) and to the underlying asset.

[13]However, it is important to note that the portfolio value of a delta–gamma hedge can still change if the riskless interest rate changes, as time passes, and/or if volatility changes. Note that equation 19.4 can be derived by applying Itô's lemma to $C = C(S,t)$ where the underlying price, S, is assumed to follow an Itô process. Then,

$$dC = \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial t}dt + \frac{1}{z}\frac{\partial^2 C}{\partial S^2}(dS)^2$$

which is (19.4), when the 'greeks' are substituted for $\partial C/\partial S$, $\partial C/\partial t$, and $\dfrac{\partial^2 C}{\partial S^2}$.

[14]Rubinstein (1985) employs actual transaction data and bid–asked spreads, accounts for dividends and early exercise, and considers several different option pricing models in his tests of option market efficiency. He finds that implied volatilities for out-of-the-money calls grow significantly higher, the shorter the time to expiration. He also finds that the market began to overprice at the money calls with a short time to expiration in his later subperiod, October 1977–September 1978. The empirical phenomenon may be explained in one of two ways: (a) all our option pricing models are incorrect, or (b) investors like to play "long shots" that offer a small chance of a large percentage payoff, and their purchase of options that offer this feature results in higher prices for these options.

[15]Another common clause is that the yearly adjustment can never exceed a preset amount. For example, while a variable-rate mortgage might specify a cap of 12.5% and a floor of 5.5%, it might also restrict the yearly change in the interest rate to be no more than 200 basis points. From Example 19.8, this would mean that one year after the loan is originated, the new interest rate could range between 6.5 and 10.5%.

[16]The payoff to the cap or floor is often negotiated to be discounted [i.e., the present value of $NP(R - R_c)(d/360)$].

[17]In foreign exchange transactions, a zero-cost collar is called a **range forward**.

[18]Instead of trading caps, floors, and collars, financial institutions could insure the risk of their net interest rate exposure by using Eurodollar futures options, or by using Eurodollar futures contracts and dynamic hedging methods (i.e., delta hedging to dynamically create an option).

[19]By convention, the forward rate in a cap, $r(t_1t_2)$, is computed using $[1 + r(0,t_2)]^{t_2} = [1 + r(0,t_1)]^{t_1}[1 + r(t_1,t_2)(t_2 - t_1)]$. The forward rates are then said to be computed on a "money market basis". You are encouraged to use Equation (19.7) to verify the cap values, and Equation (19.8) to verify the floor values.

[20]For more details on caps, floors, and collars, see Brown and Smith (1988), Abken (1989), Boyle and Turnbull (1989), and Stapleton and Subrahmanyam (1990).

PROBLEMS

19.1 Toastcoil is a firm in an enviable position. It has a monopoly in the United States. in the business of manufacturing heating elements that are put into toasters. No other firm in the country manufactures these elements. While other firms in Europe and Japan make the products, they do not export to the United States. Toastcoil does not export its product and none of its raw materials are imported. On the surface, Toastcoil faces no currency risk. It has no revenues, expenses, assets, or liabilities that are based on any foreign currency.

But this is an erroneous view of Toastcoil's position. If the ¥/$ exchange rate rises, then Japanese manufacturers may find it profitable to enter the U.S. market. If any Eurocurrency price of the dollar rises, European manufacturers may decide to export their products to the United. States. Therefore, Toastcoil does indeed face foreign exchange risk. If the dollar rises in value, the firm's profitable monopoly position may erode from foreign competitors.

How can Toastcoil use options to protect itself against this exchange rate risk?

19.2 In the winter of 1997, several Asian currencies dropped drastically in value against the U.S. dollar. Several currencies, including the Indonesian rupiah, fell by more than 50% in value.

a. One small Indonesian firm had in an earlier year borrowed \$800 million because U.S. interest rates were lower than Indonesian interest rates. Then, in July, this firm used foreign currency options to hedge part of its dollar-denominated debt. Explain how the firm could use options for this purpose.

b. If the \$/rupiah exchange rate began to decline, how would the Indonesian firm maintain its hedge?

19.3 You are given the following information:

$S = 93$

$K = 100$

$T = 0.25$ year

$r = 7.5\%$/year

$\sigma = 0.75$/year

No dividends

a. Use the BSOPM to calculate the put and call prices. Use FinCAD and put–call parity to verify your answer.

b. Use the formulas in Section 19.1.1 to calculate values for delta, gamma, vega, rho, and theta for calls and puts. Use FinCAD to verify your answers.

19.4 Consider a call option with a strike price of $K = 90$, and 70 days until expiration. The price of the underlying asset is 86.375, and the volatility of the underlying asset is 32.7%. The riskless interest rate is 6%. The actual market price of the option is 3. You might refer to Section 19.3 to answer the following questions.

a. Explain what trades you would make to arbitrage this mispriced option, assuming that your estimate of volatility, 32.7%, is correct.

b. Suppose that the subsequent weekly prices of the underlying asset are those shown in Table 19.1. Dynamically maintain your arbitrage. Create a table similar to Table 19.2, and use the Case I price for the stock price at expiration ($S_T = 97.625$). What is the actual arbitrage profit, and how does it compare to the arbitrage profit you expected?

19.5 Suppose you have positions in the following assets:

Long or Short, Number of Options or Shares	Delta/Unit
Short 1,300 shares	1.00
Long 400 puts	−0.46
Short 150 calls	0.80
Long 72 calls	0.28

Calculate the position delta. Explain what will happen to the value of the portfolio if the stock price increases by \$1.

19.6 An options trader has several long and short positions in many puts and calls on a given stock. The computed position delta of the portfolio is −12,456. Each option is on one share of stock. How many shares of stock should be bought or sold in order to be delta neutral?

19.7 Suppose you bought 100 undervalued puts with a Δ (that reflects your estimated volatility) of −0.30. You wish to create a delta-neutral position by using in-the-money puts that are correctly valued (or overvalued) and have a Δ of −0.85. How many of the latter should you trade, and should they be bought or sold?

19.8 An options market maker has the following positions in options:

Long or Short Number of Options	Put/Call	Option Delta
Long 20	Calls	0.2
Long 48	Calls	0.6
Short 16	Calls	0.82
Long 120	Puts	−0.12
Short 30	Puts	−0.51

a. What is his position delta? How can we use trades in the underlying shares, to go home delta neutral?

b. Suppose instead that he anticipates a weak opening in this stock and wants to have a position delta of −0.50 overnight. How many shares of stock should he buy or sell to achieve this?

19.9 An investor wishes to enter into a delta-neutral position with two options that, given the current price of the underlying asset, have the following prices and deltas:

Option	Price	Delta
A	14	−0.4300
B	14	+0.3300

a. Suppose an investor writes eight contracts of option A. To be delta neutral, how many contracts of option B should be traded? Should they be bought or sold?

b. The investor estimates that if the price of the underlying asset increased by one dollar, the delta of option A would become −0.4200, and the delta of option B would become 0.3340. Using this information, estimate the position gamma for the delta-neutral position you formed in part a.

c. If the underlying asset does increase in value by \$1, and the investor wishes to reestablish the delta-neutral position using the underlying asset, how many shares must he buy or sell?

19.10 Suppose $S=K_1=\$90$, $r=6\%$, $\sigma=50\%$, $T=90$ days, and $K_2=110$. Then, by using the BSOPM or the FinCAD function aaBS, fill in the following table:

	Call Option		
Variable	$K=90$	$K=110$	**Stock**
Price			
Delta			
Gamma			

a. Suppose an investor writes 400 of the 90 calls. What positions does he need in the stock and the 110 calls to be delta–gamma neutral?

b. Instead, suppose the investor buys 250 of the 90 calls. What positions does he need in the stock and the 110 calls to be delta–gamma neutral?

19.11 In Section 19.4.4.2, our computations yielded 1.22 as the initial ratio of calls written per call purchased needed to create a delta-neutral position. If the stock price remained unchanged, what will happen to that ratio as the expiration date nears? Provide an intuitive answer (i.e., do not compute new deltas). If the stock price were to rise immediately after entering the neutral spread, what would happen to that ratio? Again, do not compute new deltas. Provide an intuitive answer.

19.12 Consider the following information. Suppose an 18-month variable-rate loan exists with reset dates on March 31, June 30, September 30, and December 31. The reset rate is LIBOR. Let today be February 1, 2001. The notional principal is $25 million. The volatility of interest rates is 15%.

Loan Reset Date	Time to Expiration (years)	$r(0,t1)$	$r(t1,t2)$	Cap Value for R_c= 8%	Floor Value for R_F= 3%
03/31/01		0.035			
06/30/01		0.042			
09/30/01		0.048			
12/31/01		0.053			
03/31/02		0.055			
06/30/02		0.057			
09/30/02		0.058			

Calculate the dollar value of the cap and the floor. Use the FinCAD function aaRcapBL_fs_statv to check your solution.

19.13 Use the FinCAD function aaRcap BL_fs_statv to compute the theoretical value of the 5% floor in the example shown in Section 19.5.4.

PART 5

DERIVATIVE FRONTIERS